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COMMENT

QUATERNIONIC, COMPLEX, DUPLEX AND REAL CLIFFORD ALGEBRAS

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We show that the introduction of a new linear space, here called duplex numbers, introduces a more useful classification and representation scheme for Clifford algebras. Complex, duplex and simple double numbers belong to a ring of double numbers.

The Representation of Clifford Algebras.

For the construction and representation of Clifford Algebras we have well known general theorems and propositions (see for example Porteous 1981, Topological Geometry (2nd. ed.), Cambridge, U.K., pags. 247 and following, here denoted by $\mathcal{P}$). In particular the real algebras $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ are considered as the basic building blocks for matrix representations $\mathbb{K}(2^d)$, where $\mathbb{R}$ denotes the real field, $\mathbb{C}$ the complex field, $\mathbb{H}$ the (non-commutative) quaternion field and $\mathbb{K}(n)$ the field of square $n \times n$ matrices with an element of the $\mathbb{K}$ field at each n^2 entry.

It is well known that the constructions follow general propositions. Let us analyze them (see chapter 13 in $\mathcal{P}$, we keep Porteous numbering):

Prop. 13.17. Let X be an $\mathbf{A}$-linear space, where $\mathbf{A} = \mathbb{K}$ or ${}^2\mathbb{K}$ and $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and let S be an orthonormal subset of End X of type (p, q), generating End X as a real algebra. Then the set of matrices (see last section!)

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} : a \in S \right\} \cup \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

is an orthonormal subset of End X^2 of type $(p+1,\, q+1)$, generating End $X^2 \cong$ End $X \otimes_{\mathbf{R}} \mathbb{R}(2)$ as a real algebra.

Cor. 13.18. For each finite n, the endomorphism algebra $\mathbb{R}(2^n)$ is a universal Clifford algebra for the neutral non-degenerate space $\mathbb{R}^{n,n}$. That is, $\mathbb{R}_{n,n} \cong \mathbb{R}(2^n)$.

Proof. By induction. The basis is that $\mathbb{R}$ is a universal Clifford algebra for $\mathbb{R}^{0,0}$, and the step is Prop. 13.17.

Theorem 13.19. (Existence theorem).

Every finite-dimensional orthogonal space has a universal Clifford algebra.

Prop. 13.20. Let S be an orthonormal subset of type $(p+1, q)$ generating an associative algebra A. Then, for any $a \in S$ with $a^2 = 1$, the set

$$\{ba : b \in S\backslash\{a\}\} \cup \{a\}$$

is an orthonormal subset of type $(q+1, p)$ generating A.

Cor. 13.21. The universal Clifford algebras $\mathbb{R}_{p+1,q}$ and $\mathbb{R}_{q+1,p}$ are isomorphic.

Prop. 13.22. For $q \leq 4$, $\mathbb{R}_{0,q}$ is isomorphic, respectively, to $\mathbb{R}, \mathbb{C}$, $\mathbb{H}$, ${}^2\mathbb{H}$, or $\mathbb{H}(2)$.

It is enough, in each case, to exhibit an orthonormal subset of the appropriate type with the product of its members, in any order, not equal to 1 or -1, for each algebra has the correct real dimension, namely 2^p. Appropriate orthonormal subsets are

$\emptyset$	for	$\mathbb{R}$
$\{i\}$	for	$\mathbb{C}$
$\{i,\, k\}$	for	$\mathbb{H}$

$$\left\{ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}, \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix} \right\} \text{ for } {}^2\mathbb{H}$$

$$\text{and } \left\{ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}, \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \text{ for } \mathbb{H}(2).$$

This completes the construction of the algebras $\mathbb{R}_{p,q}$, for $p+q \leq 4$. In particular, since $\mathbb{R}_{0,1} \cong \mathbb{C}$, $\mathbb{R}_{3,0} \cong \mathbb{R}_{1,2} \cong \mathbb{C}| \otimes \mathbb{R}(2) \cong \mathbb{C}(2)$, while $\mathbb{R}_{3,1} \cong \mathbb{R}_{2,2} \cong \mathbb{R}(4)$. Theoretical physicists call $\mathbb{R}_{3,0}$ the *Pauli algebra* and $\mathbb{R}_{3,1}$ the *Dirac algebra*.

Instead of the construction of $\mathbb{R}_{p+1,q}$ from $\mathbb{R}_{p,q}$ we can sometimes use the inverse procedure. Well known are the sequences of even subalgebras going from complex-Dirac $\mathcal{D}_c$ to complex-Pauli $\mathcal{P}_c$ to complex $\mathcal{C}$ and their real subalgebras. Table I.

Finally, a more sophisticated result, leading to the 'periodicity theorem':

Prop. 13.23. Let $S = \{e_i : i \in 4\}$ be an orthonormal subset of type (0,4) of an associative algebra A with unity 1 and let R be an orthonormal subset of type (p,q) of A such that each element of S anticommutes with every element of R. Then there exists an orthonormal subset R' of type (p,q) such that each element of S commutes with every element of R'. Conversely, the existence of R' implies the existence of R.

Proof. Let $a = e_0e_1e_2e_3$ and let $R' = \{ab : b \in R\}$. Since a commutes with every element of R and anticommutes with every element of S and since $a^2 = 1$, it follows at once that R' is of the required form. The converse is similarly proved.

Cor. 13.24. For all finite p, q,

$$\mathbb{R}_{p,q+4} \cong \mathbb{R}_{p,q} \otimes \mathbb{R}_{0,4} \cong \mathbb{R}_{p,q} \otimes \mathbb{H}(2).$$

For example

$$\mathbb{R}_{0,5} \cong \mathbb{C} \otimes \mathbb{H}(2) \cong \mathbb{C}(4),$$
$$\mathbb{R}_{0,6} \cong \mathbb{H} \otimes \mathbb{H}(2) \cong \mathbb{R}(8),$$
$$\mathbb{R}_{0,7} \cong {}^2\mathbb{H} \otimes \mathbb{H}(2) \cong {}^2\mathbb{R}(8),$$

and

$$\mathbb{R}_{0,8} \cong \mathbb{H}(2) \otimes \mathbb{H}(2) \cong \mathbb{R}(16).$$

Cor. 13.25. *(The periodicity theorem).*

For all finite p, q,

$$\mathbb{R}_{p,q+8} \cong \mathbb{R}_{p,q} \otimes \mathbb{R}(16).$$

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By putting together Prop. 13.22, Prop. 13.12, Prop. 13.17, Prop. 13.20, and these last two corollaries, we can construct any $\mathbb{R}_{p,q}$. Table I shows them all, for $p+q \leq 8$. The vertical pattern is derived from Prop. 13.17, and the horizontal symmetry about the line with equation $p-q=1$ is derived from Prop. 13.20.

Table II exhibits each of the universal Clifford algebras $\mathbb{R}_{p,q}$ as the real algebra of endomorphisms of a right **A**-linear space V of the form $\mathbf{A}^m$, where $\mathbf{A} = \mathbb{R}, \mathbb{C}$, $\mathbb{H}$, ${}^2\mathbb{R}$ or ${}^2\mathbb{H}$.

When $\mathbb{K}$ is a double field (${}^2\mathbb{R}$ or ${}^2\mathbb{H}$), the $\mathbb{K}$-linear spaces $V(1,0)$ and $V(0,1)$ are called the *(real) half-spinor spaces* or *spaces of (real) half-spinors*, the endomorphism algebra of either being a non-universal Clifford algebra of the appropriate orthogonal space.

Complex Clifford Algebras

The real field may be replaced throughout the above discussion by any commutative field-in particular by the field $\mathbb{C}$. The notation $\mathbb{C}_n$ will denote the universal complex Clifford algebra for $\mathbb{C}^n$ unique up to isomorphism.

Prop. 13.28. For any $n, p, q \in \omega$ with $n = p+q, \mathbb{C}_n \cong \mathbb{R}_{p,q} \otimes_{\mathbf{R}} \mathbb{C}$, $\cong$ denoting a real algebra isomorphism.

Cor. 13.29. For any $\mathbb{R} \in \omega, \mathbb{C}_{2k} \cong \mathbb{C}(2^k)$ and $\mathbb{C}_{2k+1} \cong {}^2\mathbb{C}(2^k)$.

The *complex spinor* and *half-spinor spaces* are defined analogously to their real counterparts.

It is quite noticeable that the series $p-q=+1$ has a matrix dimension double of the matrix representing the Clifford algebras of the same $p+q$. The same situation happens with the isomorphic $p-q=-3$ or $p-q=+5$ or in general for the $\{(p+2m)-(q-2m)=+1;\ m=\pm 1, \pm 2, \ldots\}$ series of a given $n=p+q$.

The reason for this behaviour is that no need has been felt to consider composite numbers in the ring structure on the set of $\mathbb{R}^2$ other than ${}^2\mathbb{R}$ and the complex numbers field $\mathbb{C}$. This is however questioned below.

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Real Pairs, Complex and Duplex Numbers.

Consider now the ring ${}^2\mathbb{R}$. First there is the ring product of $\mathbb{R}$ with itself. Addition, defined, for all (a,b), $(c,d) \in \mathbb{R}^2$, by the formula $(a,b)+(c,d) = (a+c, b+d)$ is an abelian group structure with zero (0,0); multiplication, defined by the formula $(a,b)(c,d) = (ac, bd)$, is both commutative and associative, with unity (1,1); and there is an injective ring map $\mathbb{R} \to \mathbb{R}^2;\ \lambda \Longrightarrow (\lambda,\lambda)$ inducing a multiplication

$$\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2;\ (\lambda,(a,b)) \Longrightarrow ((\lambda,\lambda),(a,b)) \Longrightarrow (\lambda a, \lambda b).$$

This ring will be denoted by ${}^2\mathbb{R}$. Though $\mathbb{R}$ is a field, the ring ${}^2\mathbb{R}$ is not. For example, the element (1,0) does not have an inverse.

${}^2\mathbb{R}$ can be generated by the 2×2 matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, the multiplicative unity is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and its use as a basis for a Clifford algebra starts by considering the ring $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, such that $\mathbf{e}_1^2 = \mathbf{1}$. Real pairs are simple double numbers.

The **complex** numbers (a,b) with sum

$$(a,b)+(c,d) = (a+c, b+d)$$

and product

$$(a,b)(c,d) = (ac-bd, ad+bc)$$

has the unity $(1,0)$ and any non-zero element (a,b) has an inverse $((a^2+b^2)^{-1}a, -(a^2+b^2)^{-1}b)$. Then the bijective ring map

$$\mathbb{R}^2 \to \mathbb{C}; (a,b) \Longrightarrow a+ib, \text{ where } i^2 = -1$$

is used, a is called the real part and ib the "imaginary" part. A complex number maps into a twodimensional $\mathbb{R}^{2,0}$ space-space vector.

Now in the same sequence we will call **duplex** $\mathbb{D}$ the ring generated by "decreeing" $(a,b)+(c,d) = (a+c, b+d)$ and $(a,b)(c,d) = (ac+bd)(ad+bc)$.

The duplex numbers can be mapped into the form

$$\mathbb{R}^2 \to \mathbb{D};\ (a,b) \Longrightarrow a + \kappa b$$

where $\kappa^2 = 1$, a is called the real part and κb the "virtual" part. A duplex number maps into a spacetime vector, with a the spacelike part and b the time like part, we will abound on this below.

Duplex numbers $\mathbf{d}$ have duplex conjugate $\mathbf{d}^+$:

$$(a+\kappa b)^+ \to a - \kappa b$$

from the multiplication rule we can define the "absolute value" $\mathbf{dd}^+$:

$$\begin{aligned}(a+\kappa b)(a+\kappa b)^+ &= (a+\kappa b)^+(a+\kappa b)\\ &= (a+\kappa b)(a-\kappa b) = a^2 - b^2 \in I\!R\end{aligned}$$

A non null duplex number is defined $\mathbf{dd}^+ \neq 0$.

Non null duplex numbers $\mathbf{d}$ have inverse $\mathbf{d}^{-1}$

$$\mathbf{d}^{-1} = \frac{\mathbf{d}^+}{\mathbf{dd}^+},$$

then duplex numbers form a commutative ring and field. But the square of a duplex number is not a real number unless it is pure real or pure virtual. Duplex numbers have exponential maps $\mathbf{d} = fe^{\kappa g}$ which are hyperbolic transformations on a spacetime plane. The duplex numbers correspond to a map of an $I\!R^{1,1}$ spacetime with basis vectors $\mathbf{e}_0$ and $\mathbf{e}_1$, $\mathbf{e}_0^2 = -\mathbf{e}_1^1 = 1$ by multiplying by $\mathbf{e}_1$ (or $\mathbf{e}_0$) $\{\mathbf{e}_0, \mathbf{e}_1\} \to \{\mathbf{e}_0\mathbf{e}_1, 1\}$, $\mathbf{e}_0\mathbf{e}_1$ has the properties of κ.

The product of a duplex number and a complex number will be defined

$$(a+ib)(c+\kappa d) = ac + icb + \kappa ad + i\kappa bd,$$

it has therefore four degrees of freedom, each term commuting with all others. For the virtual-imaginary part $i\kappa = \kappa i$ is required.

An important property of duplex numbers is that there are idempotents:

$$P_+ = \frac{1}{2}(1+\kappa) \text{ or } P_- = \frac{1}{2}(1+\kappa)$$

$$P_+ + P_- = \mathbf{1},\ P_+ - P_- = \kappa,\ P_+P_+ = P_+,\ P_+P_- = 0,\ P_-P_- = P_-.$$

which can in fact be taken as the basis of duplex numbers: two, both null and nonzero, numbers whose sum is $\mathbf{1}$ but their product vanishes. That would have been the definition of duplex numbers in the spirit of the renaissance algebrists. With the introduction of the duplex numbers to have the set $\mathbb{K} = \{\mathbb{R},\mathbb{C},\mathbb{D},\mathbb{H}\}$, the table of the $\mathbb{R}_{p,q}$ algebras is transformed into Table III.

Remark 1. It may be thought that we can attempt a matrix representation

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then

$$\kappa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in {}^2\mathbb{R}$$

but that would not be useful if we want to keep a simultaneous non reducible representation of the real and imaginary part of a complex number and a simultaneous cummutativity of i and κ and if we demand $i\kappa$ to have a non trivial, faithful, representation.

Here we should notice that the $+1$ column has the form $\mathbb{D}(S)$ where $S = integer\left(\frac{p+q}{2}\right)$ and that in general all algebras have the form $\mathbb{R}_{p,q} \to \mathbb{K}'(S)$, with $\mathbb{K}' = \{\mathbb{R},\mathbb{C},\mathbb{D}\}$ or $\mathbb{R}_{p,q} \to \mathbb{H}\left(\frac{S}{2}\right)$. Also ${}^2\mathbb{H}(n) \subset \mathbb{D}\,\mathbb{C}(2n)$.

The matrix representations $\mathbf{M}(\mathbb{R}_{p,q})$ however are in all cases $S \times S$ matrices for the nondegenerate representations. We should remember that degenerate representations are of the form $\mathbb{R}_{p,q} \to \mathbb{K}(S^1_\kappa) \otimes \mathbb{K}(S^1_\kappa) \otimes \ldots \otimes \mathbb{K}(S^1_\kappa)$

with $S^1_\kappa = s/m$, m being the dimension of the matrix needed to represent the field $\mathbb{K}$, $m = 1$ for $\{\mathbb{R},\mathbb{C},\mathbb{D}\}$ and $m = 2$ for $\mathbb{H}$. That is: the dimension of the matrices needed to represent $\mathbb{R}_{p,q}$ will always be $S \times S$ matrices $S =$ integer $\left(\frac{p+q}{2}\right)$.

We have them the two, proven by construction, theorems:

Theorem 1. Construction of Clifford Algebras.

All universal Clifford algebras $\mathbb{R}_{p,q}$ can be constructed as outer products of members of the set $\mathbb{K}(S)$ and are related to each other as products or quotients by members of the same set $\mathbb{K} = \{\mathbb{R},\mathbb{C},\mathbb{D},\mathbb{H}\}$.

Theorem 2. Representation of Clifford Algebras.

All universal Clifford algebras have a matrix representation $\mathbf{M}(\mathbb{R}_{p,q})$ by the $S \times S$ square matrices, $S =$ integer $\left(\frac{p+q}{2}\right)$, corresponding to $\mathbb{K}(S/m_\kappa)$. m_κ

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being the dimension of the matrix needed to represent the ring $\mathbb{K}$, $m_\kappa = 1, 2$ for $\mathbb{K} = \{\mathbb{R}, \mathbb{C}, \mathbb{D}\}, \{\mathbb{H}\}$.

We see that the introduction of the duplex ring $\mathbb{D}$ allows a compact formulation of the construction and representation of Clifford algebras. The matrix representation is in particular $\mathbf{M}(S)$ with S depending only on the dimension of the algebra and not in its signature. The duplex ring $\mathbb{D}$ is interesting in itself but appears even more important in connection with the construction and representation of the multivector Clifford algebras.

The Complex, Duplex and Double Numbers Ring

The second possible choice for the commutation relation between the imaginary number $\mathbf{i}$ and the virtual number $\underline{\kappa}$ would be

$$\mathbf{i}\underline{\kappa} = -\underline{\kappa}\mathbf{i} \tag{$C-2$}$$

which will then be taken to define the basis of the double numbers $\mathbf{e} = \mathbf{i}\underline{\kappa}$. Then $\mathbf{e}^2 = \mathbf{1}$ and

$$\mathbf{i}\underline{\kappa} = \mathbf{e} \text{ or } \mathbf{ei} = k \text{ and } \mathbf{e}\underline{\kappa} = \mathbf{i}.$$

This ring is a basis for the Clifford algebra $\mathbb{R}_{1,1}$ of $\mathbb{R}^{1,1}$. The choice of the commutation relation $C-2$ puts the rings $\mathbb{C}$, $\mathbb{D}$ and ${}^2\mathbb{R}$ into one algebra. The representation of this algebra by matrices would be the 2×2 real matrices set

$$\mathbf{i} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \underline{\kappa} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\mathbf{e} \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{1} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This ring and its representation brings us to the begining of this comment (see 13.17 above) and changes the representation of all Clifford algebras into real matrices as far as $\mathbb{C}$ and $\mathbb{D}$ (or ${}^2\mathbb{R}$) have been chosen to be represented by real number matrices only. Conversely we can think that $\mathbf{e}$ should have been considered as a basic number (a basis for the double numbers) from the start.

As $\mathbf{e}^2 = \underline{\kappa}^2$ they can be interchanged for some representation purposes.

There are in this algebra possibilities of special combinations of the basis:

i. A nilpotent combination $\underline{\nu} = \mathbf{i} + \underline{\kappa}$ with the property

$$\underline{\nu}^2 = 0$$

which generates the simplest degenerate algebra, together with the unit, which is sometimes referred to as parabolic algebra.

ii. A combination of the type $\mathbf{R}_+ = \mathbf{1} + \mathbf{e}$ and $\mathbf{R}_- = \mathbf{1} - \mathbf{e}$ which split the simple pairs into two real parts.

Complex, duplex and simple double numbers (or simple real pairs) form then the ring of double numbers, a noncommutative ring. The theorems 1 and 2 above are enriched because the outer products of double numbers are the basic building blocks of Clifford algebras.

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