SPINORS, TWISTORS, SCREWS, MEXORS AND THE MASSIVE SPINNING ELECTRON

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Abstract. Spinors correspond to a representation with spin 1/2 of the Lorentz group $\mathcal{L}$ and to a trivial, invariant, representation of the translations of the full Poincaré group $\mathcal{P}$, whereas twistors also correspond to a representation with spin 1/2 of the Lorentz group but to a faithful representation of the translations group and therefore they are indeed the simplest geometrical objects which can be used in the construction of a physical theory in spacetime. We generalize twistors to the concepts of screws; we then present a one particle theory for the electron based on this wider concept.

CONTENTS

1. TWISTORS AS GEOMETRIC OBJECTS IN SPACETIME

We follow here our presentation in [Keller, 1995; 1997] where we consider complex spacetime which does nothing more than increasing the vector space by one basis vector: $C\ell_{1,3} \to C\ell_{1,3} \otimes C \approx C\ell_{0,5} \approx C\ell_{2,3} \approx C\ell_{4,1}$ and then $R^{1,3}$ is an hypersurface in the de Sitter space by fixing only one space parameter dimension.

A basic idea is that instead of complex spacetime we should start from a more fundamental, complex, formalism such that points and the spacetime continuum

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will be derived concepts. The basic objects were called twistors. Once they are defined, we have shown that Twistors correspond to well defined geometrical objects in spacetime. They can be completely described as members of the multivector algebra of spacetime and then in terms of the standard Clifford algebra $C\ell_{1,3}$ corresponding to the space $R^{1,3}$.

The inverse logical derivation is that they can be used to generate the idea of spacetime [Penrose, 1965; 1967; 1968; 1971; 1972; 1974; 1980; Robinson and Trautman, 1962; 1967; 1972; 1976; 1986; Penrose and Ward, 1980; Robinson, 1961; Hugget, 1982]. The fundamental mathematical objects should be conformally invariant (on this respect [Liu and Keller, 1996]).This space is the projective twistor space constructed out of null geodesics and their ("half") complexification (see below) to obtain the three dimensional complex projective space CP^3. A point in this space is defined by the pair $(W^A, \Pi_{A'})$ which is required to obey (here A and A' are Pauli spinor indexes, $Z^{AA'}$ and $X^{AA'}$ are (complex and real) four dimensional vectors in the Pauli algebra, that is $X = X^\mu \sigma'_\mu = X^{AA'}\xi_A \xi^\dagger_{A'}$; where $\{\sigma'_\mu\}$ are the Pauli matrices with $(\sigma'_i)^2 = -\mathbf{1}$, $\sigma_0 = \mathbf{1}$ and the $\xi_A(\xi^\dagger_A)$ are Pauli basis spinors (transpose spinors))

$$\begin{aligned} W^A &= iZ^{AA'}\Pi_{A'} \\ \text{and} \qquad Z^{AA'} &= Z_0^{AA'} + \lambda^A \Pi^{A'} \end{aligned}$$

with the complex vector $Z_0^{AA'}$ and the Pauli spinor $\Pi^{A'}$ fixed and λ^A variable; $\Pi_{A'}$ is conjugated spinor to $\Pi^{A'}$. If we consider now a real spacetime point (in spinor notation) $X^{AA'}$ then the projective twistor $(iX^{AA'}\Pi_{A'}\ ,\ \Pi_{A'})$ defines a real null geodesic through $X^{AA'}$ in the direction $\Pi^A\Pi^{A'}$ as described below. But the motivation of the twistor theory is to derive the concept of spacetime point. Twistor theory was afterwards used to describe zero rest mass fields [Ward and Wells, 1990] and subsequently to construct spacetime from some (deformed) twistor space.

Let us first show the geometric nature of the twistors, they are members of the (complex) Clifford algebra of spacetime.

Consider a Dirac spinor ψ. We know that it is a member of the minimum left ideal of the geometric algebra $C\ell_{1,3}$ of spacetime $R^{1,3}$ generated by the Clifford algebra of the set of four vectors γ_μ

$$\begin{gathered} \mathbf{1},\ \gamma_\mu,\ \gamma_\mu\gamma_\nu = g_{\mu\nu} + \gamma_{\mu\nu}, g_{\mu\nu} = diag(\mathbf{1}, -\mathbf{1}, -\mathbf{1}, -\mathbf{1}) = g_{\nu\mu}, \\ \gamma_{\mu\nu} = -\gamma_{\nu\mu}, \gamma_\rho\gamma_{\mu\nu} = g_{\rho\mu}\gamma_\nu - g_{\rho\nu}\gamma_\mu + \gamma_{\rho\mu\nu} \\ \\ \text{and} \\ \gamma_\lambda\gamma_{\mu\nu\rho} = g_{\lambda\mu}\gamma_{\nu\rho} - g_{\lambda\nu}\gamma_{\mu\rho} + g_{\lambda\rho}\gamma_{\mu\nu} + \gamma_5 \quad \text{or} \quad \gamma_5 = \gamma_{0123}, \\ \text{all} \quad \{\mu, \nu, \lambda, \rho\} = 0, 1, 2, 3. \end{gathered} \tag{1}$$

This defines, out of the antisymmetric product of the set of four basis vectors γ_μ, the basic multivectors $\gamma_{\mu\nu\ldots}$, totally antisymmetric in the interchange of the vector indexes $\mu, \nu, \ldots$, here represented by several vector (greek) indexes quantities. The number of indexes define the grade, or blade, of basis multivectors. Multivectors have a representation in terms of the Dirac spinors $\gamma_{\mu\nu\ldots} \to (\gamma_{\mu\nu\ldots})^{aa'}\psi_a\psi^\dagger_{a'}$. The Dirac spinors themselves have four (complex) degrees of freedom $\Psi \to \Psi^a\psi_a$, where the ψ_a are basis spinors and $a = 1, 2, 3, 4$. Below we show that we can use the correspondence: $(a = 1) \to (R, \uparrow)$, $(a = 2) \to (R, \downarrow)$, $(a = 3) \to (L, \uparrow)$ and $(a =$

4) → $(L, \downarrow)$, that is right (R) and left (L) handed spinors of spin up $(\uparrow)$ and down $(\downarrow)$, or chiral representation.

All elements $\mathbf{M}_p \in \mathbf{M} = \{\gamma_\mu, \gamma_{\mu\nu}, \gamma_{\lambda\mu\nu}, \gamma_5; \mathbf{1}, i\mathbf{1}\}$ of the (complex) Clifford algebra, called (complex) multivectors, such that $\mathbf{M}_p^2 = \mathbf{M}_p\mathbf{M}_p = \mathbf{1}$ can be used to construct projectors $\mathbb{P}_p = \frac{1}{2}(\mathbf{1} + \mathbf{M}_p), \mathbb{P}_{-p} = \frac{1}{2}(\mathbf{1} - \mathbf{M}_p)$ with $\mathbb{P}_p\mathbb{P}_{-p} = 0$ and $\mathbb{P}_p + \mathbb{P}_{-p} = \mathbf{1}$. The combination of four commuting projectors $\{\mathbb{P}_p, \mathbb{P}_{-p}, \mathbb{P}_q, \mathbb{P}_{-q}\}$ suffices to classify the Dirac spinors

$$\psi = \psi_{pq} + \psi_{p(-q)} + \psi_{(-p)q} + \psi_{(-p)(-q)} \tag{2}$$

where $\psi_{ab} = \mathbb{P}_a\mathbb{P}_b\psi = \mathbb{P}_{ab}\psi$, given that

$$\mathbb{P}_{pq} + \mathbb{P}_{p(-q)} + \mathbb{P}_{(-p)q} + \mathbb{P}_{(-p)(-q)} = \mathbf{1} \tag{3}$$

There are two $\mathbf{M}_p$ (complex, or in fact imaginary) commuting unit multivectors $i\gamma_5$ and $i\gamma_{12}$ (that is $(i\gamma_5)^2 = (i\gamma_{12})^2 = \mathbf{1}$ and $\gamma_5\gamma_{12} = \gamma_{12}\gamma_5$) which are very convenient for analysis of spinors; they generate the index correspondence mentioned above for handedness and for spin [Keller, 1984; 1991].

Now the construction generated by the projectors $\mathbb{P}_R = \frac{1}{2}(1 + i\gamma_5)$ and $T_\mathbf{x} = 1 + \gamma_5\mathbf{x}$, with the position vector $\mathbf{x} = \mathbf{x}^\mu\gamma_\mu$, $\mu = 0, 1, 2, 3$ applied to a spinor ψ, is called a reference twistor $\eta_\mathbf{x}$ [Keller, 1995; 1997]associated to $\mathbf{x}$ and ψ

$$\eta_\mathbf{x} = T_\mathbf{x}\mathbb{P}_R\psi \quad \text{or} \quad \eta_\mathbf{x} = (\mathbf{1} + \gamma_5\mathbf{x})\mathbf{P_R}\psi = (\mathbf{1} + \boldsymbol{\gamma_5}\mathbf{x})\boldsymbol{\Pi}, \quad \text{here} \quad \boldsymbol{\Pi} = \mathbf{P_R}\psi. \tag{4}$$

Π is a right handed Dirac spinor which can be represented by the couple of a Pauli ξ (usually called Weyl) spinor and zero $\Pi \to \left(\begin{smallmatrix} 0 \\ \xi \end{smallmatrix}\right)$.

The transpose twistor, starting from $\bar\psi = \psi^\dagger\gamma_0$ and considering $\mathbb{P}_R = \mathbb{P}_L^\kappa$ (where $M^\kappa = (\sum_A a^A\gamma_A)^\kappa = \sum_a (a^A)^*\gamma_A$ the * indicating complex conjugation) is

$$\bar\eta_{\bar{\mathbf{x}}} = \bar\psi\mathbb{P}_L(\mathbf{1} + \gamma_5\bar{\mathbf{x}}) = \bar\Pi(\mathbf{1} + \gamma_5\bar{\mathbf{x}}) \text{ product for } \mathbf{x} \text{ real} \tag{5}$$

is such that the scalar product

$$\bar\eta_\mathbf{x}\eta_\mathbf{x} = \bar\Pi\Pi + 2\bar\Pi\gamma_5\mathbf{x}\Pi + x^2\bar\Pi\Pi = 2\bar\Pi\gamma_5\mathbf{x}\Pi, \qquad \text{because} \tag{6a}$$

$$\bar\Pi\Pi = \bar\psi\frac{1}{2}(1 - i\gamma_5)\frac{1}{2}(1 + i\gamma_5)\psi = 0. \tag{6b}$$

That is, it represents the expectation value of the (dual of the) position $\gamma_5\mathbf{x}$ with respect to spinor Π. The outer product

$$\eta_\mathbf{x}\bar\Pi = (\mathbf{1} + \gamma_5\mathbf{x})\Pi\bar\Pi = (\mathbf{1} + \gamma_5\mathbf{x})q \tag{7}$$

here $q = \Pi\bar\Pi$ is the right-handed part of a null vector $Q = \psi\bar\psi$ (a single outer product of Dirac spinors or of Weyl spinors can only correspond to fixed handedness null vectors), that is the presence of

$$q = \mathbb{P}_R\psi\bar\psi\mathbb{P}_L, \tag{8}$$

gave origin to the Penrose [Penrose, 1965; 1967; 1968; 1971; 1972; 1974; 1980; Robinson and Trautman, 1962; 1967; 1972; 1976; 1986; Penrose and Ward, 1980; Robinson, 1961; Hugget, 1982] interpretation of a twistor as a composite of a null vector q and a “flag” $\gamma_5\mathbf{x}q$, as far as a bivector represents an oriented surface. The multivector corresponding to the twistor is

$$\underline{\eta}_{\mathbf{x}} = \eta_{\mathbf{x}}\bar{\Pi} = q + \gamma_5\mathbf{x}q = q + i\mathbf{x}q. \tag{9}$$

it contains the projection $Q \to q = I\!P_R Q I\!P_L$. In (9) we have $\gamma_5\mathbf{x} = -\mathbf{x}\gamma_5$ and $-\gamma_5 q = iq$ because q is a right-handed projection.

A supermatrix representation of the above relations, although superfluous, is very helpful to visualize the different structures. The vectors

$$\gamma_\mu \to \left\{ \gamma_0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix};\ i = 1,2,3 \right\} \tag{10}$$

here the σ_i are the positive square Pauli matrices $(\sigma_i)^2 = \mathbf{1}$, $\sigma_1\sigma_2\sigma_3 = i\mathbf{1}$ and $\sigma_i\sigma_j = i\sigma_k$; $i,j,k = 1,2,3$ cyclic.

The hypervolume $\gamma_5 = i\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} = \sigma_1\sigma_2\sigma_3\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$, the adjoint spinor $\bar{\Pi} = (\mathbf{\Pi}^*, 0) = (\Pi_1^*, \Pi_2^* 0, 0)$, then the twistor

$$\eta_{\mathbf{x}} = \begin{pmatrix} i\vec{x}\mathbf{\Pi} \\ \mathbf{\Pi} \end{pmatrix} \qquad \text{remark} \qquad \mathbf{\Pi} = I\!P_R \eta_x, \tag{11}$$

where $\mathbf{x} = \begin{pmatrix} 0 & \vec{x} \\ \vec{x}^c & 0 \end{pmatrix}$, and $\vec{x}^c$ is the left-handed quaternion conjugate of $\vec{x}$. The use of the multivector $\underline{\eta}_{\mathbf{x}}$ or of the twistor spinor $\eta_{\mathbf{x}}$ can be done indistinctly. ψ can also be replaced by a multivector as discussed in the paper on the electron theory in the first chapter of this issue.

We then see that the admitted geometrical interpretation considers, as a result, not the vector $\mathbf{x}$ but its three dimensional projection $\vec{x}$ and moreover the factor $i = \sigma_1\sigma_2\sigma_3$ is the three dimensional volume element.

In the context of Clifford algebras, [Crumeyrolle, 1990]twistors have been discussed in length by [Ablamowicz *et al.,* 1982].

2. An Analysis of the Use of Spacetime Clifford Algebra for a Representation of the Poincaré Group

2.1. Representation of the Poincaré Group as a Chiral Group in Spacetime Algebra

The equations describing physical relations in spacetime (or in ordinary three dimensional space) should be covariant under both Lorentz transformations $I\!L$ and changes of origin d of the coordinate system. This set of operations, called the Poincaré group, of which rotations and changes of origin in ordinary three dimensional space are subgroups, is characterized by the pair $\{I\!L, d\}$.

The group product is $\{\mathbb{L}_2, d_2\}\{\mathbb{L}_1, d_1\} = \{\mathbb{L}_3, d_3\}$. In the geometric algebra of spacetime $R^{1,3}$ (the Clifford algebra denoted $R_{1,3}$ or $C\ell_{1,3}$) a position vector $\chi_0 = \chi^\mu \gamma_\mu$ is transformed (d is a vector and $\mathbb{L}_n$ the exponential of a bivector)

$$\chi_0 \to \chi_1 = \mathbb{L}_1 \chi_0 \tilde{\mathbb{L}}_1 + d_1 \to \chi_2 = \mathbb{L}_2 \chi_1 \tilde{\mathbb{L}}_2 + d_2 \tag{12}$$

$$\text{or} \quad \chi_2 = \mathbb{L}_2 \mathbb{L}_1 \chi_0 \tilde{\mathbb{L}}_1 \tilde{\mathbb{L}}_2 + \mathbb{L}_2 d_1 \tilde{\mathbb{L}}_2 + d_2 \tag{13}$$

$$\chi_2 = \mathbb{L}_3 \chi_0 \tilde{\mathbb{L}}_3 + d_3 \tag{14}$$

defining (the tilde operation reverses the product of two multivectors)

$$\mathbb{L}_3 = \mathbb{L}_2 \mathbb{L}_1 \text{ and } d_3 = \mathbb{L}_2 d_1 \tilde{\mathbb{L}}_2 + d_2. \tag{15}$$

The "multiplication" of the Poincaré group is well defined but cumbersome. There are several representations, some of which are reasonable to handle. For example the use of the matrix form (here the $\mathbb{L}_i$ are square matrices and the χ_i and d_i column matrices, the 0^ℓ are row matrices)

$$\begin{gathered} \begin{pmatrix} \mathbb{L}_1 & d_1 \\ 0^\ell & \mathbf{1} \end{pmatrix} \begin{pmatrix} \chi_0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbb{L}_1 \chi_0 + d_1 \\ 1 \end{pmatrix} \text{ and} \\ \begin{pmatrix} \mathbb{L}_2 & d_2 \\ 0^\ell & \mathbf{1} \end{pmatrix} \begin{pmatrix} \chi_1 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbb{L}_2 \chi_1 + d_2 \\ 1 \end{pmatrix} \end{gathered} \tag{16}$$

or

$$\begin{pmatrix} \mathbb{L}_3 & d_3 \\ 0^\ell & \mathbf{1} \end{pmatrix} \begin{pmatrix} \chi_0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbb{L}_2 & d_2 \\ 0^\ell & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbb{L}_1 & d_1 \\ 0^\ell & \mathbf{1} \end{pmatrix} \begin{pmatrix} \chi_0 \\ 1 \end{pmatrix} \quad \text{then} \tag{17}$$

$$\begin{pmatrix} \mathbb{L}_3 & d_3 \\ 0^\ell & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbb{L}_2 \mathbb{L}_1 & \mathbb{L}_2 d_1 + d_2 \\ 0^\ell & \mathbf{1} \end{pmatrix} \tag{18}$$

clearly shows that the "product" of group elements are elements of the group. Group multiplication is matrix multiplication here.

In geometric algebra there is a representation of the elements of the group which allows geometric multiplication as the group (non abelian) multiplication.

$$\{\mathbb{L}_2, d_2\}\ \{\mathbb{L}_1, d_1\} = \{\mathbb{L}_3, d_3\}$$

For this, geometric algebra representation, we use the product of the elements $(1+\varepsilon d)$ and $\mathbb{L}$ which, separately have as group multiplication, the geometric product

$$\mathbb{L}_3 = \mathbb{L}_2 \mathbb{L}_1 \text{ and } (1 + \varepsilon d_3) = (1 + \varepsilon d_2)(1 + \varepsilon d_1), \tag{19}$$

where $d_3 = d_2 + d_1$ and $\varepsilon d_2 \varepsilon d_1 = 0$ requiring that either $\{\varepsilon^2 = 0,\ \varepsilon d = d\varepsilon\}$ or $\{\varepsilon_+ \varepsilon_- = 0,\ \varepsilon_+ d = d\varepsilon_-\}$. In the first case $\varepsilon^2 = 0$ is a nilpotent operation commuting with the vectors d. In the second case $\varepsilon_+(\varepsilon_-)$ is a projector operator

$$\varepsilon_+ \varepsilon_- = \varepsilon_- \varepsilon_+ = 0,\ \varepsilon_+ + \varepsilon_- = 1, \tag{20}$$

which can be written in terms of a unit multivector e, $e^2 = \mathbf{1}$ which, $ed = -de$, anticommutes with the vectors d. The $\varepsilon_+ = \frac{1}{2}(1 + e)$ and $\varepsilon_- = \frac{1}{2}(1 - e)$. In general a suitable $\{\varepsilon; \varepsilon^2 = 0,\ \varepsilon d = d\varepsilon\}$ or $\{e; e^2 = 1,\ ed = -de\}$ can only be found in an algebra of a dimension higher than the Clifford algebra $R_{p,q}$ corresponding to the space $R^{p,q}$. The formal definition of ε or e is enough for the purpose of studying the Poincaré group but the possibility of physical usefulness or insight would be lost.

In the Dirac algebra $\mathcal{D}$ corresponding to $R_{1,3}$ one usually admits its complexification, corresponding to the use of $R_{0,5} \simeq R_{2,3} \simeq R_{4,1}$, that is $\mathcal{D} = \{R_{1,3} \oplus iR_{1,3} \simeq R_{0,5}\}$. The commonly used operators $i\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ and $i\gamma_{12} = i\gamma_1\gamma_2$ are good examples of this complexification of the algebra. It is $i\gamma_5$ which has the property $i\gamma_5\gamma_\mu = -\gamma_\mu i\gamma_5$ and $(i\gamma_5)^2 = 1$ required for the use in (20) and (21). Then the group of translations Γ has two isomorphic representations $\Gamma : d \to d'$, which are $(1 + \mathbb{P}_R d)$ and $(1 + \mathbb{P}_L d)$ where

$$\begin{array}{c} \mathbb{P}_R = \tfrac{1}{2}(1 + i\gamma_5) \text{ and } \mathbb{P}_L = \tfrac{1}{2}(1 - i\gamma_5) \\ \mathbb{P}_R + \mathbb{P}_L = 1,\ \mathbb{P}_R\mathbb{P}_L = \mathbb{P}_L\mathbb{P}_R = 0 \quad \text{and} \\ \mathbb{P}_R\mathbb{P}_R = \mathbb{P}_R,\ \mathbb{P}_L\mathbb{P}_L = \mathbb{P}_L,\ \mathbb{P}_R d = d\mathbb{P}_L. \end{array} \tag{21}$$

Here the $\mathbb{P}_R$ and $\mathbb{P}_L$ are the operators for right handedness and left handedness projection respectively: $\psi = \mathbb{P}_R\psi + \mathbb{P}_L\psi = R_0 + L_0$.

The Lorentz transformations $\mathbb{L} = exp(\theta^{\mu\nu}\gamma_{\mu\nu}/2)$ by angles $\theta^{\mu\nu}$ in the planes $\gamma_{\mu\nu}$ act

$$\mathbb{L} : \chi \to \chi' = \mathbb{L}\xi\tilde{\mathbb{L}}\,,\ \mathbb{L}\tilde{\mathbb{L}} = \tilde{\mathbb{L}}\mathbb{L} = 1, \tag{22}$$

$$\text{do form a multiplication group } \mathbb{L}_3 = \mathbb{L}_2\mathbb{L}_1\,,\ \tilde{\mathbb{L}}_3 = \tilde{\mathbb{L}}_1\tilde{\mathbb{L}}_2 \tag{23}$$

which can be used to construct, together with the $(1 + \varepsilon d)$ a set of two representations of the Poincaré group.

The representations of the Poincaré group are

$$\mathbb{P}_i^{(L)} = (1 + \mathbb{P}_L d_i)\mathbb{L}_i\ ;\ \mathbb{P}_i^{(R)} = (1 + \mathbb{P}_R d_i)\mathbb{L}_i \tag{24}$$

and ($A = L, R$ and, reminder, $\mathbb{L}_j\mathbb{P}_L = \mathbb{P}_L\mathbb{L}_j$ and $\mathbb{P}_L d_n = d_n\mathbb{P}_R$)

$$\begin{array}{c} \mathbb{P}_k^{(A)} = \mathbb{P}_j^{(A)}\mathbb{P}_i^{(A)} = (1 + \mathbb{P}_L d_j)\mathbb{L}_j(1 + \mathbb{P}_L d_i)\mathbb{L}_i, \\ \mathbb{P}_k^{(A)} = (1 + \mathbb{P}_L(d_j + \mathbb{L}_j d_i \mathbb{L}_j))\mathbb{L}_j\mathbb{L}_i, \\ \mathbb{P}_k^{(A)} = (1 + \mathbb{P}_L d_k)\mathbb{L}_k, \\ \text{also} \quad (1 + \mathbf{P}_{\mathrm{L}}\mathrm{d_k})(1 - \mathbf{P}_{\mathrm{L}}\mathrm{d_k}) = 1. \end{array} \tag{25}$$

The invertible operators $(1+\mathbb{P}_A d)$ become then an important part of the study of the invariances and the symmetries related to relativistic quantum theory of particle and interaction fields. Their use to construct a new representation of the Poincaré Lie algebra is thus both straightforward and clarifies the reason for some features of the theory of elementary particles. The relations (25) could have been the best starting point for the instruction of a theory of twistors. We should remember that any spinor can be written $\psi = \mathbb{L}\psi_0$ and that $\mathbb{L}\mathbb{P}_L = \mathbb{P}_L\mathbb{L}$.

When we work in the realm of the Clifford algebras the complexification of a space corresponds to the increase of only one dimension in the space of basis vectors. Basis vectors should all anticommute among themselves. Other procedures, besides complexification, like duplexification or doubling [Keller, 1994a]play a similar, and equivalent role. This should be very clear when discussing twistor spaces.

3. The Development of Some Concepts of the Spacetime Twistor Program in Terms of Geometric Spacetime Algebra

Physical events are related to other physical events by light rays (or at least by assumed trajectories corresponding to null vectors), an important condition then is the intersection of light rays and of the light ray emanating (supported) by another light ray. We have seen at the beginning that twistors correspond to a special combination of a vector $\mathbf{X}$ and a null vector $\mathbf{q}$. We can consider several possibilities: a) $\mathbf{X}$ and $\mathbf{q}$ null vectors, b) $\mathbf{X}$ general vector and $\mathbf{q}$ null vector or c) $\mathbf{X}$ and $\mathbf{q}$ general vectors. In all cases we can consider the complexification of the vectors. We start by considering the conformally invariant product of two twistors.

3.1. Twistors (Double Fibration)

An important concept, basic for the understanding of twistor techniques, is the definition of *double fibration.* Let us, with [Ward and Wells, 1990], consider $\mathbb{P}$ a projective twistor space, $\mathbf{M}$ a compactified complexified Minkowski space and $\mathbb{F}$ the correspondence space between $\mathbb{P}$ and $\mathbf{M}$, the double fibration diagram, using the projection mappings: $\mu(p,m) = p$ and $\nu(p,m) = m$

$$\begin{tikzcd}
& \mathbb{F} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\
\mathbb{P} & & \mathbf{M}
\end{tikzcd}$$

where $p \in \mathbb{P}$ and $m \in \mathbf{M}$ describe the transformation of information from $\mathbb{P}$ to $\mathbf{M}$, the real Minkowski space is realized as a subset of the complexified Minkowski space $\mathbf{M}$. The correspondence $p = \mu \bullet \nu^{-1}(m)$ defines a *geometric transformation* from the complex vector m to the twistor p or vice versa

$$m = \nu \bullet \mu^{-1}(p)$$

be conformally invariant [Liu and Keller, 1996].This space is the projective twistor space constructed out of null geodesics and their ("half") complexification (see below) to obtain the three dimensional complex projective space CP^3. A point in this space is defined by the pair $(w^A, \Pi_{A'})$ which is required to obey

$$w^A = iZ^{AA'}\Pi_{A'} \quad \textit{and } Z^{AA'} = Z_0^{AA'} + \lambda^A \Pi^{A'} \tag{26}$$

with $Z_0^{AA'}$ and $\Pi^{A'}$ fixed and λ^A variable.

If we consider now a real spacetime point (in spinor notation) $X^{AA'}$ then the projective twistor $(iX^{AA'}\Pi_{A'}, \Pi_{A'})$ defines a real null geodesic through $X^{AA'}$ in the direction $\Pi^A\Pi^{A'}$ as described below. But the motivation of the twistor theory is to derive the concept of a spacetime point. Twistor theory was afterwards used to describe zero rest mass fields [Ward and Wells, 1990] and subsequently to construct spacetime from some (deformed) twistor space.

The idea is to pull back cohomology groups from $\mathbb{P}$ to $\mathbb{F}$ and then to push it down to $\mathbf{M}$ to solutions of field equations. This procedure allows the determination of new solutions of the field equations, classified by the group of the cohomologies of $\mathbb{P}$.

Let us now obtain the scalar products corresponding to what Penrose calls the twistor invariant, the product of two twistors when the transpose is taken considering that we can define $\gamma_5^\dagger = -\gamma_5$ (as is obvious from the representation (10) of γ_5 in terms of gamma matrices). Then, instead of (5) we define the adjoint $\hat{\eta}$ as follows

$$\hat{\eta} = \bar{\psi}\mathbb{P}_R(1 + \gamma_5^\dagger\bar{\mathbf{x}}) = \bar{\Pi}(1 - \gamma_5\bar{\mathbf{x}}) \tag{27}$$

and consider the product of two twistors

a) Corresponding to the same spinor ψ and to the same vector real $\mathbf{x}$

$$J^{\psi\psi}_{\mathbf{xx}} = \hat{\eta}_{\mathbf{x}}\eta_{\mathbf{x}} = \bar{\Pi}\gamma_5(\mathbf{x} - \mathbf{x})\Pi = 0. \tag{28}$$

This is obviously invariant to multiplication of $\eta_{\mathbf{x}}$ by a complex factor, then from the eight degrees of freedom (four complex numbers) only six are geometrically significant and (28) reduces them to five real parameters. These parameters can be chosen to correspond too to the direction of the null ray (light trajectory) and to its intersection with a base hyperplane $t = 0$.

b) Corresponding to the same spinor ψ, but to two different vectors $\mathbf{x}$ and $\mathbf{x}'$

$$J^{\psi\psi}_{\mathbf{xx}'} = \bar{\Pi}\gamma_5(\mathbf{x}' - \bar{\mathbf{x}})\Pi \tag{29}$$

which will be zero if it corresponds to a point where two (real $\bar{\mathbf{x}} = \mathbf{x}'$) light rays intersect. This is what could correspond to a definition of a point from twistors. For variable $\mathbf{x}'$ and fixed $\mathbf{x}$ we define a congruence usually called Robinson congruence.

Let us now consider the multivector twistors $\underline{\eta}_{\mathbf{x}}$ and their generalization. Reminder

$$\underline{\eta}^{(\psi)}_{\mathbf{x}} = \eta_{\mathbf{x}}\bar{\Pi} = \mathbf{q} + \gamma_5\mathbf{x}\mathbf{q} \tag{30}$$

where $\mathbf{q}$ is a null vector considered (by definition not included in (5)) supported at the position $\mathbf{x}$. See analysis at the end. The obvious generalization is, both $\mathbf{q}$ and $\mathbf{x}$, to be allowed to become arbitrary (complex) vectors.

4. SCREWS AND MULTIVECTOR SCREWS

Let us analyze the possibilities open. The need for $\mathbf{q}$ to be a null vector stemmed from its introduction as a single outer product of a spinor $\mathbf{\Pi}$ and its conjugate $\bar{\mathbf{\Pi}}$. On the other hand $\mathbf{x}$ could have been null or general.

$$\underline{\eta}^{(\psi)}_{\mathbf{x}} = (1 + \gamma_5\mathbf{x})\mathbf{q} \to \underline{\boldsymbol{\eta}}_{\mathbf{x},\mathbf{y}} = (1 + \gamma_5\mathbf{x}\mathbb{P}_A)\mathbb{P}_A\boldsymbol{Y} \tag{31}$$

where $\mathbb{P}_A = \{\mathbb{P}_L, \mathbb{P}_R\}$ is either one of the idempotent left or right handedness projectors, in particular we could have defined $\mathbf{q} = \mathbb{P}_L\boldsymbol{Y}\gamma_0$ for $\boldsymbol{Y}$ such that $\mathbf{q}$ a null vector. But in (31) we want to allow $\boldsymbol{Y}$ to be a general (complex vector)

$$\boldsymbol{Y} = \boldsymbol{Y}^{ab}\xi_a\bar{\xi}_b \qquad a, b = 1, 2, 3, 4 \tag{32}$$

where ξ_a (and $\bar{\xi}_b$) are a general basis set for Dirac spinors (and adjoint Dirac spinors). $\boldsymbol{Y}^{ab} \in \mathbb{C}$. If $\mathbf{q} = \mathbb{P}_L\boldsymbol{Y}\gamma_0 = \Pi\bar{\Pi}$ then $\boldsymbol{Y} = \mathbb{P}_\uparrow\boldsymbol{Y}$, (remember that up and down are only relative, otherwise free, directions).

The presence of γ_0 in the definition of $\mathbf{q}$ from $\boldsymbol{Y}$ is necessary to connect with (5) and (30) because $\mathbf{q}$ corresponds to a spacetime "cut" of a vector $\mathbf{q_v}$ (a spacetime "cut" is the multiplication of a multivector by a time vector γ_0)

$$\mathbf{q} = \mathbf{q_v}\gamma_0 = q^\mu \gamma_\mu \gamma_0 = q^\mu \Sigma_\mu . \tag{33}$$

Where the spacetime quaternions $\Sigma_\mu = \gamma_\mu \gamma_0 = \gamma_{\mu 0}$ can be represented by block diagonal matrices, the same as $\mathbf{q}$, with σ_μ in the main diagonal, while the γ_μ and the particular time vector γ_0 were represented in (10) by block diagonal matrices with σ_μ in the second diagonal. $\Sigma_\mu = \gamma_\mu \gamma_0 \to \begin{pmatrix} \sigma_\mu & 0 \\ 0 & -\sigma_\mu \end{pmatrix}$; $\mu = 1, 2, 3$ while $\Sigma_0 = \mathbf{1}$. In (31) we find all elements of the Dirac (complex spacetime) algebra. The $\boldsymbol{Y}$ are odd (so is $\mathbb{P}_L \boldsymbol{Y}$) and the $\mathbf{X}\boldsymbol{Y}$ are even (the same as $\gamma_5 \mathbf{X} \mathbb{P}_L \boldsymbol{Y}$), the presence of γ_5 makes all elements, upon which it acts, become their dual: scalars to pseudoscalars, vectors to trivectors and space-space bivectors to space-time bivectors. $\underline{\boldsymbol{\eta}}_{\mathbf{x},\mathbf{y}}$ is then a full (complex) multivector.

Here is where the discussion of part C, about the full Poincaré group is now directly relevant. The factors $(1 + \gamma_5 \mathbf{X} \mathbb{P}_A)$ are representations of the translations group

$$(1 + \gamma_5 \mathbf{X} \mathbb{P}_A)(1 + \gamma_5 \mathbf{X}' \mathbb{P}_A) = (1 + \gamma_5 (\mathbf{X} + \mathbf{X}') \mathbb{P}_A). \tag{34}$$

Reminder: $\gamma_5 \mathbf{X} \mathbb{P}_A \gamma_5 \mathbf{X}' \mathbb{P}_A = \gamma_5 \mathbf{X} \gamma_5 \mathbf{X}' \mathbb{P}_B \mathbb{P}_A = 0$, where $B \neq A$. These factors also commute with the rotations

$$\begin{aligned} R(1 + \gamma_5 \mathbf{X} \mathbb{P}_A) \mathbb{P}_A \boldsymbol{Y} R^{-1} &= (1 + \gamma_5 R \mathbf{X} R^{-1} \mathbb{P}_A) \mathbb{P}_A R \boldsymbol{Y} R^{-1} \\ &= (1 + \gamma_5 \mathbf{X}'' \mathbb{P}_A) \mathbb{P}_A \boldsymbol{Y}'', \end{aligned} \tag{35}$$

where $\mathbf{X}'' = R(\mathbf{X})$ and $\boldsymbol{Y}'' = R(\boldsymbol{Y})$ are the rotated vectors.

That is: the vector $\mathbf{X}$ is a **position** (frame fixed) vector whereas $\boldsymbol{Y}$ is a frame free (sometimes called just **"free"**) vector. $\boldsymbol{Y}$, or in the original twistor Π, is to be acted on by rotations but not by translations. Then it should represent a physical phenomena and $\boldsymbol{Y}$ is not a position vector. In geometry $\boldsymbol{Y}$ is a vector that should be associated with magnitude or direction, not with position.

If a twistor or our new objects, in fact a special combination of a vector and its product with a position vector, should become a field over $\mathbf{x}$, it is the chiral part $\mathbb{P}_A \boldsymbol{Y} = \boldsymbol{Y}_A$ that should become $\boldsymbol{Y}_A(\mathbf{x})$. Even in the case where both $\boldsymbol{Y}$, $\mathbf{x} \in \mathbb{C}^4$. That is the free vector (representing a vectorial magnitude field) is the part that carries the position dependence and $\mathbf{x}$ is the part that anchors the field.

In the use of the (multivector) twistors an integration over $\mathbf{x}$ averages the field and an integration over $\boldsymbol{Y}$ averages over the (auxiliary) field.

We can moreover take the imaginary phase out of the definition of the multivector pair, represented by γ_5 in (4) and (31), and keep the pair we will call screw $S_{\mathbf{x},\mathbf{y}}$

$$S^{(A)}_{\mathbf{x},\mathbf{y}} = (1 + \mathbf{X} \mathbb{P}_A) \boldsymbol{Y} \tag{36}$$

The screws are also faithful, nontrivial, representations of the Poincaré group. The factor $(1 + \mathbf{X} \mathbb{P}_A)$ is invertible (the factor $(1 + \gamma_5 \mathbf{X} \mathbb{P}_A)$ was also invertible) and the factor $\boldsymbol{Y}$ can also be invertible. Consider

$$(1+\mathbf{X}\mathbb{P}_A)(1-\mathbf{X}\mathbb{P}_A)=1 \tag{37}$$

because $\mathbf{X}\mathbb{P}_A\mathbf{X}\mathbb{P}_A=\mathbf{X}^2\mathbb{P}_B\mathbb{P}_A=0$, B is $B\neq A$, also

$$\boldsymbol{Y}\boldsymbol{Y}^{-1}=\boldsymbol{Y}(\boldsymbol{Y}/\boldsymbol{Y}^2)=1 \qquad \text{if } \boldsymbol{Y}^2\neq 0. \tag{38}$$

A uniform screw would be $S^{(A)}_{\mathbf{x},\mathbf{y}}$ where $\boldsymbol{Y}=\boldsymbol{Y}(\mathbf{x})=\mathbb{R}(\mathbf{X})Y_0\mathbb{R}^{-1}(\mathbf{X})$ with $\mathbb{R}(\mathbf{X})$ a rotation generator linear in $\mathbf{X}$. In Clifford algebra rotations are generated by the bivectors $\gamma_{ij}(i,j=1,2,3)$, their general form is $\mathbb{R}(\mathbf{X})=\exp(\frac{1}{2}\sigma^{ij}(\mathbf{X})\gamma_{ij})$. The uniform rotation would be a linear dependence on some $\mathbf{X}_1=a\mathbf{X}_0$, such that $\sigma^{ij}=a\sigma_1^{ij}+\sigma_0^{ij}$. Then the screw will correspond to rotations in the plane γ_{ij} proportional to the displacement of the vector $\mathbf{X}$ according to the scalar product $(\mathbf{X}\mathbf{X}_0)_{\text{scalar}}$. We could also consider the screw to describe a helical path. The new objects are suitable candidates for robotics, vision analysis or models of angular momentum carrying objects. Here we will consider an electron field to be represented by combinations of screws. In (32) we can of course consider that Y is a multivector, all multivectors having the same decomposition, as mentioned below (1), given by (32). The considerations of this paragraph applying equally to all multivectors given that rotations are multivector grade concerning functions. A multivector spinor will be used to represent an electron in the next section.

A final, geometrical, consideration. If $\mathbf{X}$ is a position vector and $\mathbf{q}$ a null vector, only the line where $\mathbf{q}$ is embedded can be known because we do not really know $\mathbf{X}$ but only the product $\mathbf{X}\mathbf{q}$ and this product is unchanged is we replace $\mathbf{X}\rightarrow\mathbf{X}+\alpha\mathbf{q}$, the scalar part is increased in $\alpha(q)^2_s=0$ and the bivector part in $\alpha q\wedge q=0$. Only if $\mathbf{X}$ itself is null then $\mathbf{X}\wedge\mathbf{q}/\mathbf{X}\cdot\mathbf{q}=\tan\frac{\theta}{2}$ and we can know the point of "support" of the null vector $\mathbf{q}$ by the null vector $\mathbf{X}$.

This is the geometrical model of the original twistors, null rays supported by null rays.

In the generalization, if we consider non-null vectors supported by non-null vectors, the product has a scalar part $X^\mu Y_\mu$ which changes if we change $\mathbf{X}\rightarrow\mathbf{X}+\alpha\boldsymbol{Y}$ because no longer $\alpha(\boldsymbol{Y})^2$ will be zero, then the product $\mathbf{X}\boldsymbol{Y}$ and the knowledge of $\boldsymbol{Y}$ determines $\mathbf{X}$ completely. We can, from the analysis of the generalized twistor find the support position vector $\mathbf{X}$ and the supported free vector $\boldsymbol{Y}$. The twistor η (see (4)) has a dual η^D. We have seen that η has 4 complex components, they may be labeled $\eta_\alpha,\ \alpha=1,2,3,4,\ \eta^0$ also has 4 operator components $\eta^\alpha,\ \alpha=1,2,3,4$ and, being a function and a conjugated operator pair, they obey the commutation relations

$$[\eta^\alpha,\eta_\beta]=\delta^\alpha_\beta \tag{39}$$

useful to construct field theories.

Dual twistors correspond to the operator pair $\left(-i\,\frac{\partial}{\partial Z^{AA'}}\,\frac{\partial}{\partial \pi^{A'}}\ ,\ \frac{\partial}{\partial \pi^{A'}}\right)$.

5. Multivector Screw for the Electron and its Spacetime Position X Vector

Let us now present the form in which a multivector screw can contain all the information we know to be necessary to describe an electron at the single particle level.

First let us remind the physical content and the geometrical content of the (multivector) wave function ψ. A free electron wave function ψ_0 can be written (see Keller 1997, pag. 3 of this issue) in the well known form (see Casanova 1976 for a discussion)

$$\psi_0 = \sqrt{\rho}\, e^{\beta\gamma_5} R_0 \tag{40}$$

where ρ is the statistical weight ($\sqrt{\rho}$ is the amplitude) of the wave function at a point X, then the probabilistic nature of the wave function is contained in ρ. The factor $e^{\beta\gamma_5}$ with β the Takabayashi angle is the way to determine if we are describing a particle or its antiparticle and the rotor R_0 describes a rotation in the spin plane γ_{12} corresponding to intrinsic angular momentum of the particle's field at point X

$$R_0 = e^{\gamma_{12} p_0 X/\hbar} \tag{41}$$

Then this factor, the rotor R_0, contains the first quantization properties of the electron and, simultaneously, its spin in the arbitrary $e_1 - e_2$ plane described by γ_{12}. An arbitrary wave function would be $\psi = R\psi_0$ with R a Lorentz transformation which can be taken to be $R(X)$ to describe the gauge interactions of the electron. We have described elsewhere how R can also describe the full electroweak interaction and not only the electromagnetic and gravitational parts [Keller, 1991 and page 3 in this issue].

Now the position vector of the electron X and the normalized spin vector Y of the electron can be incorporated in the full complex multivector screw M

$$M = (1 + i(X + iX_I))\,\psi = (1 + iZ)\psi. \tag{42}$$

We propose to call this complex multivector MEXOR or the Multivector to represent the Electron and its position X spacetime vecTOR. (We have also chosen this name for obvious cultural reasons!) and it is the geometric object which completely represents a massive, interacting, spinning electron as will be analyzed in the next section.

But first let us correct with the already known concepts of Dirac spinor and of twistors.

The Dirac spinor basis φ_i, $i = 1, 2, 3, 4$ and the transpose $(\varphi^\dagger)^i$ obey

$$(\varphi^\dagger)^i \varphi_j = \delta^i_j\ , \tag{43}$$

they can generate the multivectors as

$$M = M^j_i \varphi_j (\varphi^\dagger)^i \tag{44}$$

in particular $\psi = \psi_k(\varphi^\dagger)^k$ with $k = \{a, b\}$ in eq. (2) above and we can represent $\psi_k = \begin{pmatrix} \xi \\ \pi \end{pmatrix}$ where ξ and π are Weyl spinors and conjugated spinors respectively. And conversely any multivector can be projected into a Dirac spinor φ_k

$$\begin{aligned} M \to M_k &= M\varphi_k = M^j_i \varphi_j (\varphi^\dagger)^i \varphi_k \\ &= M^j_i \varphi_j \delta^i_k = M^j_k \varphi_j \end{aligned} \tag{45}$$

(see Keller and Rodríguez [1992]).

Then the multivector ψ contains in fact four components ψ_k which actually correspond, if we take the chiral basis representation, to the right handed spin up or down and to the left handed spin up or down components. In a 4 × 4 matrix representation of ψ each column ψ_k corresponds to each one of this four possibilities as a reference. And, from the use of the chirality projectors (4)

$$\mathbb{P}_L + \mathbb{P}_R = \mathbf{1}, \quad \mathbb{P}_R\mathbb{P}_L = \mathbb{P}_L\mathbb{P}_R = 0, \quad \mathbb{P}_R^2 = \mathbb{P}_R \text{ and } \mathbb{P}_L^2 = \mathbb{P}_L \tag{46}$$

we can write the MEXOR M in (45) as the multivector twistor pair

$$M = (1 + iZ)\psi = (\mathbb{P}_L + \mathbb{P}_R + iZ(\mathbb{P}_L + \mathbb{P}_R))\,(\mathbb{P}_L + \mathbb{P}_R)\psi = M_L + M_R \tag{47}$$

and then for each basis spinor φ_k; $k = 1, 2, 3, 4$ to project $\psi \to \psi_k = \psi\varphi_k$ as the four sets

$$\begin{aligned} &M_k = (1 + iZ\mathbb{P}_L)\mathbb{P}_L\psi_k + (1 + iZ\mathbb{P}_R)\mathbb{P}_R\psi_k \quad \text{or} \quad M_k = (M_k)_L + (M_k)_R \\ &\text{with } M = M_k(\varphi^\dagger)^k; \end{aligned} \tag{48}$$

and each M_k can describe an electron as a sum of two twistors in reference to each of the four Dirac spinors ψ_k for the electron being given.

6. A Comprehensive One Particle Theory for the Electron Based on Mexors

Once the MEXOR corresponding to an electron has been defined we can use the analysis of twistors in section 3 above to show that the formulation in terms of mexors contains some key new features for the determination of the theory of the electron.

We will show that the Dirac equation with mass $m \neq 0$ for ψ_k is a consequence of having introduced the MEXOR. This is related to the fact that (45) is not Poincaré covariant directly but only its separated left and right handed parts M_L and M_R are faithful representations of the Poincaré group.

The twistor product of the M_k with itself and with its adjoint has then several terms which have different meanings: $(M_k)^\dagger_L\gamma_A(M_k)_L$, $(M_k)^\dagger_R\gamma_A(M_k)_R$, $(\tilde{M}_k)_L\gamma_A(M_k)_R$, $(\tilde{M}_k)_L\gamma_A(M_k)_L$, $(\tilde{M}_k)_R\gamma_A(M_k)_R$ and $(\tilde{M}_k)_R\gamma_A(M_k)_L$.

All of them related however to the analysis of section 3 above. We form combinations of the type of bilinear covariants and invariants with the form $\tilde{M}_k\gamma_A M_k$, related to the Dirac theory with the γ_A the hermitian operator obtained from the Clifford algebra (1).

The γ_A will contain three types of information. First that information related to geometric quantities: γ_μ for vector quantities like momentum or $\gamma_{\mu\nu}$ for bivector quantities like angular momentum or electromagnetic field strengths; second the twistor or spinor "metrics": ε_{AB} is given by δ_1 and the twistor metric by γ_1, the twistor conjugation is given by $i\gamma_5\gamma_0$ represented by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and third the projection which selects from the spinor or from the twistor the desired components, for example given $\eta_x = \begin{pmatrix} iX\mathbf{\Pi} \\ \mathbf{\Pi} \end{pmatrix} \to I\!\!P_R\eta_x = \Pi = I\!\!P_R\psi = \begin{pmatrix} 0 \\ \mathbf{\Pi} \end{pmatrix}$ and then $I\!\!P_R$ selects the basic spinor from the twistor (in usual twistor notation $Z^\alpha = \left(W^A, \pi_{A'}\right) \to I\!\!P_R Z^\alpha = (0, \pi_{A'})$ and $I\!\!P_L Z^\alpha = (W^A,\ 0)$.

First remember that because in the twistors we have by definition

$$V^A = iZ^{AA'}\xi_{A'}, \quad \text{and} \quad W^A = iZ^{AA'}\pi_{A'} \tag{49}$$

from which we can form the invariant, using the twistor metric $m_{\alpha\beta}$, $w_0 = m_{\alpha\beta}V^\alpha W^\beta = \bar\pi\xi$ and also we can extract the complex vector Z from the twistors and the representation of the γ^μ basis vectors of the Clifford algebra

$$Z^\mu = i\omega_0^{-1}(\bar V\delta^\mu\pi - \bar W\delta^\mu\xi) = M_k^\dagger\gamma^\mu M_k \tag{50}$$

$$\text{with } X^\mu = \text{ real part } Z^\mu \text{ and } X_I^\mu = \text{ imag part } Z^\mu \tag{51}$$

we also obtain Y the Pauli-Lubanski or spin spacetime vector. This is more clearly shown if we form the momentum vector

$$P^\mu = \tilde M_k\gamma^\mu M_k \tag{52}$$

resulting in

$$P^\mu = \bar\xi(\sigma^\mu)\xi + \bar\pi\sigma^\mu\pi \tag{53}$$

which is the momentum part of the Gordon decomposition of the currents. The normalization of the ψ given by (55) below; and then check that Y obeys

$$Y^\mu Y_\mu = 1, \quad Y^\mu P_\mu = 0. \tag{54}$$

Furthermore, from the existence of the cross products in (52) we find that

$$P^\mu P_\mu = 2\omega_0^*\omega_0 = m^2 > 0 \tag{55}$$

then for the determination of the wave function ψ the massive Dirac equation has to be used with the value of m resulting from (55). This is in agreement with the Higgs postulate that mass is to be obtained from the coupling of the left handed and the right handed parts of the electron's field.

The cross products of the terms with $(\psi_k)_L$ and $(\psi_k)_R$ define a couple of vectors e_1 and e_2 which together with the normalized Y and P form an orthonormal tetrad with P and Y gauge invariant but e_1 and e_2 only defined up to a rotation angle in the plane they generate.

As we can recover the spinors from the twistors $I\!\!P_R\eta_R = I\!\!P_R\psi$ and $I\!\!P_L\eta_L = I\!\!P_L\psi$, then $\psi = I\!\!P_R\eta_R + I\!\!P_L\eta_L$ and all bilinear quantities and analysis of the standard theory are recovered. It is also clear that the fundamental differential equation becomes the Dirac equation applied to ψ. The bispinor ψ can also be mapped into a spinor pair [Keller and Rodríguez Romo, 1991].

We can also map one of the two twistors, say the left-handed, into a right-handed twistor, by straightforward conjugation and then the MEXOR will result equivalent to a twistor pair as in the analysis of the massive particle with spin made by Tod [1975, 1977; Tod and Perjes 1976; Bette 1984-1989] with all their results recovered by the MEXOR approach.

Finally from the gauge freedom of the spinor $\psi \to \mathbb{R}(x)\psi$ and of the twistor $Z^{AA'} \to Z^{AA'} + \lambda^A \pi^{A'}$ we introduce through $\mathbb{R}(x)$ the gauge fields and through the results, change in Z given by the terms $\lambda^A \pi^{A'}$, describe the changes the gauge fields produce in Y and in the e_1 and e_2 vectors besides the well known changes the gauge fields produce in the momentum vector P. This can also be described as a sum of spinors

$$\psi' = \psi + (\mathbb{R}(x) - 1)\psi \tag{56}$$

and therefore as the coupling with new MEXORS

$$(1 + iZ)[(\mathbb{R}(x) - 1)\psi] \tag{57}$$

or their corresponding decomposition into a sum of (two additional) twistors.

We have described elsewhere how if the mass of the electron is given the masses of the other elementary particles are describable in terms of numbers associated to the representations on the Dirac Clifford algebra [Keller, 1994b].

The gauge approach to interactions is expressed in Clifford algebra with the use of the representations of the Lorentz transformations $\mathbb{L}$ given by (12) above. They are the $\mathbb{R}$ multivectors generated by the exponentiations of the bivectors $(\lambda \neq \{\mu \neq \nu\})\gamma_{\mu\nu}$

$$\gamma_{\mu\nu} \begin{cases} \gamma_\mu & = -\gamma_{\mu\nu}\gamma_{\mu\nu} \quad = -g_{\mu\mu}\gamma_\nu \\ \gamma_\nu & = \gamma_{\mu\nu}\gamma_\nu \qquad\ \ = g_{\nu\nu}\gamma_\mu \\ \gamma_\lambda & = \ \gamma_\lambda\gamma_{\mu\nu} \end{cases}$$

generators of the Lorentz transformations, which rotate vectors in the plane $\mu = \nu$ they represent and leave vectors orthogonal to that plane unchanged.

$$\mathbb{R} = e^{\gamma_{\mu\nu}\phi^{\mu\nu}}$$

The action on multivectors is

$$M \to \mathbb{R}M\widetilde{\mathbb{R}} \quad \text{with} \quad \mathbb{R}\widetilde{\mathbb{R}} = \widetilde{\mathbb{R}}R = \mathbf{1}$$

For example, applying $e^{\gamma_{\mu\nu}\varphi}$ with $(\gamma_{\mu\nu})^2 = -\mathbf{1}$, we obtain

$$\begin{aligned} &(\cos\varphi + \gamma_{\mu\nu}\ \mathrm{sen}\varphi)\ \gamma_\lambda(\cos\varphi - \gamma_{\mu\nu}\ \mathrm{sen}\varphi) \\ &= \gamma_\lambda \cos^2\varphi + \cos\varphi\ \mathrm{sen}\varphi(\gamma_{\mu\nu}\gamma_\lambda - \gamma_\lambda\gamma_{\mu\nu}) -\ \mathrm{sen}^2\varphi\gamma_{\mu\nu}\gamma_\lambda\gamma_{\mu\nu} \\ &= \gamma_\lambda(\cos^2\varphi H g_{\lambda\lambda} - g_{\lambda\nu} - g_{\mu\lambda})g_{\lambda\lambda}\ \mathrm{sen}^2\varphi) \\ &-\ \mathrm{sen}^2\varphi(g_{\nu\lambda}g_{\mu\mu}\gamma_\nu + g_{\lambda\mu}g_{\nu\nu}\gamma_\mu - (g_{\lambda\lambda} - g_{\lambda\nu} - g_{\lambda\mu})g_{\lambda\lambda}\gamma_\lambda) \\ &+2\cos\varphi\ \mathrm{sen}\varphi(g_{\nu\lambda}\gamma_\mu - g_{\mu\lambda}\gamma_\nu) \end{aligned}$$

corresponding to a rotation by an angle of 2φ of the vectors in the plane $\mu = \nu$. Whereas the action on spinors is

$$\psi \to \mathbb{R}\psi$$

corresponding to a rotation by an angle φ in the plane $\mu - \nu$.

These rotations and boosts change the energy-momentum of the particle represented by the field. But, moreover, they change the components of the basic geometric quantities of the theory, most relevant the local tetra $\dot{X}$, Y, e_1 and e_2, then the gauge fields are equivalent to local deformations of the reference spacetime and constitute in fact a total geometrization of the gauge fields and interactions.

The changes in the spinor are

$$\psi' - \psi = (\mathbb{R} - \mathbf{1})\psi = \mathbb{R}'\psi$$

and then the changes in the MEXOR, assuming that the vector Y is also acted by the gauge rotor $\mathbb{R}$, as $Y \to Y' = \mathbb{R} Y \widetilde{\mathbb{R}}$, is

$$(1 + iZ)\psi \to (1 + iZ')\psi'$$

with $Z' = X + iY'$ and then writing $Z' = Z + Y' - Y$

$$(1 + iZ)\psi \to (1 + iZ)\psi + (1 + iZ)\mathbb{R}'\psi + (Y - Y')\psi'$$

but in practice Y is not used and the last term drops out. This shows that in principle the gauge fields correspond to the addition of a new MEXOR resulting from the change in the spinor part. Therefore the physics of the new model is in fact a manifestation of the spinor transformation (gauge phase factors). The interaction field strengths $F^{\mu\nu}$ change the energy momentum

$$\dot{P}^{\mu} = eF^{\mu\nu}P_{\nu}$$

and induce a precession of the spin vector and an additional rotation of the spin plane

$$\begin{aligned} \dot{S}^{\mu} &= eF^{\mu\nu}S_{\nu} \\ \dot{e}_1^{\mu} &= -\omega e_2^{\mu} - eF^{\mu\nu}e_{1\nu} \\ \dot{e}_2^{\mu} &= \omega e_1^{\mu} + eF^{\mu\nu}e_{2\nu} \end{aligned}$$

where ω is the spin angular velocity $\hbar/2mc$.

This finishes this short discussion on the completeness of the MEXOR fields to describe the known features of the single particle approach to the theory of the electrons. All analysis presented in the paper Keller [1997], page 3 of this issue, applies without further changes.

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REFERENCES

Abłamowicz Rafał, Zbigniew Oziewicz and Jan Rzewuski [1982], Clifford Algebra Approach to Twistors, *J. Math. Phys.*, **23**, 231-242

Bette Andreas
- [1984] *J. Math. Phys.* **25** (8), 2456
- [1988] in Proceedings of the Conference "Microphysical reality and quantum formalism" in Urbino
- [1988] "Fundamental theories of Physics", Reidel's Monograph Seires
- [1989] *Reports on Mathematical Physics* **28**, 133-140

Crumeyrolle Albert [1990], *Orthogonal and Symplectic Clifford Algebras, Spinor Structures*, Kluwer A.P., Dordrecht

Hugget Steven [1982], "Development of twistor ideas", in: *Mathematics of the Physical Spacetime*, J. Keller ed., Facultad de Química, Universidad Nacional Autónoma de México, Publishers, México, 57-65

Keller Jaime
- [1984] *Int. J. Theor. Phys.* **23** (9), 817-837
- [1991] *Int. J. Theor. Phys.* **30** (2), 137-184
- [1994*a*] *Advances in Applied Clifford Algebras* **4** (1), 1-12
- [1994*b*] *Advances in Applied Clifford Algebras* **4** (S1), 379-393
- [1995] "Twistors as Geometric Objects in Spacetime", in: Clifford Algebras and Spinor Structures, P. Lounesto y R. Abłamowicz eds., Kluwer Academic Publishers, The Netherlands, 133-136
- [1997] Twistors and Clifford Algebras, In: "Clifford Algebras and their Applications in Mathematical Physics" Habeta Lehrstuhl II Für Mathematiks, Aachen Germany

Keller Jaime and Suemi Rodríguez-Romo [1991], *J. Math. Phys.* **31** (10), 2501-2510

Keller Jaime and Adan Rodríguez [1992], *J. Math. Phy.* **33** (1) 161-170

Liu Yu-Fen and Jaime Keller [1996], A Symmetry of Massless Fields, *Journal of Mathematical Physics* **37** (9), 4320-4332

Penrose Roger
- [1965] "Zero Rest-Mass Fields Including Gravitation: Asymptotic Behaviour" *Proc. Roy. Soc. Lond.* **A284**, 159-203
- [1967] *J. Math. Phys.* **8**, 345
- [1968*a*] "Structure of Space-Time". In: "Battelle Rencontres", C.M. De Witt B. and J.A. Wheeler, eds., Benjamin, New York, 121-235
- [1968*b*] "Twistor Quantisation and Curved Space-Time", *Int. J. Theor. Phys.*, $\underline{1}$, 61-99
- [1971] "Angular Momentum: An Approach to Combinational Space-Times", In: "Quantum Theory and Beyond" T. Bastin, ed., Cambridge University Press, Cambridge, 151-180

Penrose Roger
- [1972] "On the Nature of Quantum Geometry" In: "Magic Without Magic. J.A. Wheeler", J.R. Klander, ed., Freeman, San Francisco, 333-354
- [1974] "Relativistic Symmetry Groups" In: Group Theory in Non-Linear Problems", A.O. Barut, ed., D. Reidel, Dordrecht, 1-58
- [1980] "Null Hypersurface Initial Data for Classical Fields of Arbitrary Spin and for General Relativity". *Gen. Rel. Grav.*, **12**, 225-264

Penrose Roger and R. S. Ward [1980], "Twistors for Flat and Curved Space-Time" In: General Relativity and Gravitation, One Hundred Years After the Birth of Albert Einstein, A. Held, ed., Plenum, New York

Robinson Ivan [1961], Null Electromagnetic Fields, *J. Math. Phys.* **2**, 290-291

Robinson Ivan and Albert Trautman

[1962] Some Spherical Gravitational Waves in General Relativity *Proc. Roy. Soc. Lond.* **A265**, 463-473

[1967] Twistor Algebra *J. Math. Phys.*, **8**, 345-366

[1972] "The Geometry of Impulsive Gravitational Waves". In: General Relativity: papers in honour of J.L. Synge, L. O'Raifeartaigh, ed., Clarendon Press, Oxford, 101-115

[1976] Nonlinear Gravitons and Curved Twistor Theory *Gen. Rel. Grav.*, **7**, 31-52

[1986] *Spinors and Spacetime*, Vol. 2 "Spinor and Twistor Methods in Space-time Geometry", Roger Penrose and W. Reindler, Cambridge University Press, Cambridge

Tod K. P.

[1975] "Massive spinning particles and twistor theory", Doctoral Dissertation, Mathematical Institute, University of Oxford

[1977] *Rep. Math. Phys.* **11** (3), 339

Tod K. P. and Z. Perjes [1976], *General Relativity and Gravitation* **7** (11), 903-913

Ward R. S. and R. O. Wells [1990], "Twistor Geometry and Field Theory", Cambridge U.P., Cambridge and references therein