# EQUATIONS OF MOTION FOR SPINNING MASSIVE PARTICLES OVER TWISTOR FIELDS

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## 1. Introduction

One of the comprehensive developments of the idea of spinorization of the phase space has been achieved in the twistor theory [1]. Using the condition of masslessness the twistor approach established a deep relation between phase space variables of massless spinning particles and spinor wave functions. In this formalism the equations of motion of massless particles have been transformed into algebraic equations over oscillator ladder operators of the helicity.

The success of the twistorial description of massless particles has inspired several authors to apply this tool to construct also a twistorial representation of the phase space of massive spinning particles. Dirac's bispinors contain two independent Weyl's spinors and in that sense keeps two times more information. This induces the idea [2, 3, 4] to construct a dynamics of massive spinning systems using pairs of twistors corresponding to (two) massless particles. It

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turns out that (a double) phase space of classical massless object may serve as a building block for the construction of the irreducible phase space of a massive spinning particle. In some sense, the massive spinning particle obtained by the reduction procedure may be regarded as a bound (confined) system of two directly interacting massless spinning constituents.

The development of this approach in the quantum case requires the transformation of the Dirac equation into an algebraic system of equations over twistor variables. In the best approach to this program one has to exchange the mass parameter in the Dirac equation by two mutually complex conjugated values. This complex value arises as a simple consequence of the bispinor representation of momentum in the Dirac equation. The same result is obtained using the bispinor representation of the momentum in the basis of Dirac-gamma matrices, in that case starting from Pauli—Fierz identities, we obtain a Dirac like equation in six dimensional momentum space, two components corresponding to the complex mass parameter. (Let us note, that these components accept another interpretation if bispinors are substituted by twistor coordinates [5].

In this paper we examine both representations: the spinorial- with well known van der Waerden symbols, and the vectorial expressed in terms of Dirac-gamma matrices. In this way the bridge between the twistorial description of massive particle and the concept of screws (introduced in [6]) is established.

In Section 2, for the convenience of the reader, we simply summarize the assumptions and definitions used in the twistor's theory of massless particles.

In Section 3 we build the twistor phase space for the massive particle. We show that the twistor representation of momentum demands a modification of the Dirac equation introducing a complex mass parameter.

In Section 4 the equation of motion of Lorentz-Bargmann-Michel-Telegdi in twistorial phase space deduced in full detail.

The paper should be considered as an analysis and comparison of otherwise known formulations.

## 2. Internal Twistorial Structure of Momentum and Angular Momentum for Massless Particles

For the convenience of the reader, we simply summarize the assumptions and definitions used in the twistor's theory of massless particles. In the Minkowski space-time a particle $\mathcal{M}$ is described by the total four-momentum $p^a$, total angular momentum $M^{ab}$ and its helicity $s$. Under the action of the Poincaré group the components of the pair $(p^a, M^{ab})$ transform covariantly. Regardless of whether a particle possesses mass or not, it will be called pointlike if we have a splitting

$$M^{ab} = 2x^{[a}p^{b]} + S^{ab}, \tag{2.1}$$

with $x^a$, representing its position and with $S^{ab}$, representing intrinsic spin such that

$$S^{ab}p_b = 0. \tag{2.2}$$

The masslessness of a system is expressed by the fact that

$$m^2 := p^a p_a = 0. \tag{2.3}$$

Using these relations we obtain that the Pauli-Lubanski four-vector

$$S_a = -\frac{1}{2}\epsilon_{abcd}M^{bc}p^d. \tag{2.4}$$

is proportional to $p^a$ $S^a = sp_a$, with the factor of proportionality $s$ being the helicity of the massless object. Equations (2.1-2.4) also imply that if $s \neq 0$ then $x^a$ may be taken as any point on a null hyperplane defined by $x^a p_a = d$, where $d$ is a translationaly dependent Lorentz scalar.

To obtain a twistor formulation of these relations, corresponds to finding their (Weyl) spinor structure. The real four-momentum satisfying zero rest mass condition (2.3) can be expressed as matrix factorization of the hermitian matrix

$$P_{AA'} := p_a \sigma^a_{AA'} = \pi_A \pi_{A'}, \tag{2.5}$$

where $\pi_A$ is usually Weyl spinor. Thus the spinor $\pi_A$ carries the information on the momentum of massless particles. To give a spinor representation of the six-angular momentum one needs an addition spinor $\omega^A$ carrying the information of space-time coordinates. It is defined as

$$\omega^A := -ix^a \sigma_a^{AA'} \pi_{A'} = -iX^{AA'}\pi_{A'}, \tag{2.6}$$

where $\omega^A$ is also a 4-translation dependent Weyl spinor. Then the skew-symmetric angular momentum $M^{ab}$ is defined as ($\epsilon^{AB}$ being the spinor metric)

$$M^{AA'BB'} = -i(\epsilon^{A'B'}(\omega^A\overline{\pi}^B + \omega^B\overline{\pi}^A) - \epsilon^{AB}(\overline{\omega}^{A'}\pi^{B'} + \overline{\omega}^{B'}\pi^{A'})) \tag{2.7}$$

Its dual, $M_{ab} = \frac{1}{2}\eta_{abcd}M^{cd} = *M_{ab}$ can be written

$$M_{ab} \Rightarrow *M_{AA'BB'} = \epsilon_{A'B'}(\omega_A\overline{\pi}_B + \omega_B\overline{\pi}_A) + \epsilon_{AB}(\overline{\omega}_{A'}\pi_{B'} + \overline{\omega}_{B'}\pi_{A'}) \tag{2.8}$$

The quantities $s$ and $d$ also may be consistently expressed as functions of $\omega^A$ and $\pi_A$

$$s = \frac{1}{2}(\omega^A \overline{\pi}_A + \pi^{A'} \overline{\omega}_{A'}), d = \frac{i}{2}(\omega^A \overline{\pi}_A - \pi^{A'} \overline{\omega}_{A'}). \tag{2.9}$$

The twistor $Z^\alpha$ is directly defined as the mathematical object with components

$$Z^\alpha := (\omega^A, \pi_{A'}), \tag{2.10}$$

represented in the complexified Minkowski space-time $\mathcal{CM}$ picture as the locus of all complex points $X^{AA'}$. When the solution $X^{AA'}$ of (2.6) lies on a real null plane, $Z^\alpha$ is a null twistor. Twistor indices are raised and lowered by transposition $(\ )^{\mathrm{T}}$ and complex conjugation: $Z^\alpha$ twistor has the simple conjugate $Z_\alpha$ of components

$$Z_\alpha = (\overline{\pi}_A, \overline{\omega}^{A'})^{\mathrm{T}}, \tag{2.11}$$

so that contractions, of which the simplest representative is $Z^\alpha Z_\alpha$, are invariant under twistor transformations. This invariant has the form

$$Z^\alpha Z_\alpha = \overline{\pi}_A \omega^A + \pi_{A'} \overline{\omega}^{A'}. \tag{2.12}$$

Comparing (2.12) with (2.9) we find that the twistor norm expresses twice the value of helicity

$$s = \frac{1}{2} Z^\alpha Z_\alpha. \tag{2.13}$$

Then in twistor formalism the classical and quantum motions of a free massless and spinning relativistic particle may be described as a direct product of two Weyl spinors endowed with natural symplectic structure. This structure in twistor coordinates is given by

$$\Omega = i dZ^\alpha \wedge dZ_\alpha. \tag{2.14}$$

The relations (2.5-2.8) express the transformation of momentum and angular momentum under the map $(x^a, p_b) \to (\omega^A, \pi_{A'}) = (Z^\alpha)$ on the algebraic level. The Poincaré covariance of the pair $(\omega^A, \pi_{A'})$ implies the Poincaré covariance of the pair $(p_a, M^{ab})$. Let us now treat $(\omega^A, \pi_{A'})$ as the Poincaré- covariant coordinates of a point in four-dimensional complex vector space $T$. $T$ is a symplectic vector space equipped with global Poincaré- covariant canonically conjugate coordinates: $-i\omega^A$ and $\overline{\pi}_B$ or,equivalently, $-i\pi_{A'}$ and $\overline{\omega}^{B'}$ fulfilling the following Poincaré- covariant canonical Poisson algebra

$$\{\omega^A, \overline{\pi}_B\} = \delta^A_B, \quad \{\pi^{A'}, \overline{\omega}_{B'}\} = \delta^{A'}_{B'} \tag{2.15}$$

Within the definitions (2.5-2.7) it is a straightforward tedious task to check that the canonical Poincaré-covariant Poisson bracket relations imply the Poincaré-covariant Poisson bracket realization of the algebra

In the twistor formalism the equations of motion for massless fields accept the form of algebraic equations, given in the language of creation and annihilation operators. The role of the latter operators now play the components of twistor variables. The Weyl equation for a free massless particle for spin $\frac{1}{2}\hbar$ is given by

$$P^{AA'}\pi_A\Phi = 0, P^{AA'}\pi_{A'}\Phi = 0, \tag{2.16}$$

where $\Phi$ is the holomorphic function of $\omega^A$.

## 3. The Massive Particle and its Twistorial Phase Space

We have seen in the previous section that the twistor description of the system is immediately available if the components of momentum satisfy the masslessness condition. One should define a similar fundamental object for the description of massive systems as well. When the spinor structure of the momentum is realized the condition of masslessness is transformed into Weyl algebraic identities. With the same aim one can obtain a bispinor representation of the momentum of massive particles, satisfying the relativistic relation between momentum and mass, if the Weyl algebraic identities may be extended for the massive Dirac equations.

Let us examine the momentum structure of the massive particle for the Dirac equation

$$(\gamma^a p_a)\Psi = m\Psi. \tag{3.1}$$

Employing the chiral basis of Dirac $\gamma$-matrices where $\gamma_5$- diagonal in their van der Waerden description, we have considered as representations of basis vectors for spacetime

$$\gamma_a = \begin{pmatrix} 0 & (\sigma_a)^{AA'} \\ (\sigma_a)_{BB'} & 0 \end{pmatrix}, \tag{3.2}$$

with $(\sigma_a)_{AA'}$ the van der Waerden symbols, corresponding to Pauli matrices

$$(\sigma_0)_{AA'} = I, \quad (\sigma_k)_{AA'} = \sigma_k, k = 1, 2, 3. \tag{3.3}$$

The Dirac bispinor $\Psi$ over Weyl spinors is given by

$$\Psi = \begin{pmatrix} u_A \\ \overline{v}^{B'} \end{pmatrix}, \overline{\Psi} = \begin{pmatrix} v^B, & \overline{u}_{A'} \end{pmatrix}. \tag{3.4}$$

So, in that description the Dirac equation becomes equivalent to the set of two equations

$$\begin{aligned} P_{AA'}u^{A'} &= m\overline{v}_A \\ P^{AA'}\overline{v}_A &= mu^{A'}, \end{aligned} \tag{3.5}$$

or

$$\begin{aligned} P_{AA'}v^{A'} &= -m\overline{u}_A \\ P^{AA'}\overline{u}_A &= -mv^{A'}, \end{aligned} \tag{3.6}$$

where

$$P_{AA'} = p_a \sigma^a_{AA'}. \tag{3.7}$$

From these equations we obtain

$$P^{AA'}(\overline{v}_A\overline{u}^B - \overline{u}_A\overline{v}^B) = m(u^{A'}\overline{u}^B + v^{A'}\overline{v}^B), \tag{3.8}$$

or

$$P^{AA'} = \frac{m}{(\overline{v}_C\overline{u}^C)}(u^{A'}\overline{u}^B + v^{A'}\overline{v}^B). \tag{3.9}$$

In (3.9) the momentum of the massive particle is given in the bispinor representation which has been directly deduced from the Dirac equation. Within conventional field theory the dynamical equations of motion for the massless particles (fermions) are understood as a consequence of the limit $m \to 0$ reducing the equations of the massive particles into the equations of massless particles. One of the primary example of such transformation is the reduction of Dirac equation onto Weyl equations. In that case by putting $m = 0$ one obtains the splitting of the Dirac equation into two independent parts, where (only) one of the parts (namely left-handed) has been accepted as an equation for the neutrino. Since the other part (right-handed) for the case $m = 0$ has to disappear, it is natural to suggest, that from $m = 0$ it follows $u^A = 0$, or vice versa. Accordingly the expression for $P^{AA'}$ has to be reduced to the formula (2.5). It may be realized iff in (3.9) we put

$$m = (constant)(\overline{v}_C\overline{u}^C), \tag{3.10}$$

and thus we obtain the bispinor representation of the momentum for the massive particle

$$P^{AA'} = \pi^{A'}\overline{\pi}^A + \eta^{A'}\overline{\eta}^A. \tag{3.11}$$

This representation implies that the mass is given by

$$m^2 = p^a p_a = 2f\overline{f}, \tag{3.12}$$

with $f = (\pi^{A'} \quad \eta_{A'}), \overline{f} = (\overline{\pi}^A \quad \overline{\eta}_A)$.

Using the spinor representation (3.11) one may build two sets of identities, which after appropriate quantization of twistor variables has to give back the Dirac equations. The first set of equations is given by

$$\begin{aligned} p_a\sigma^a_{AA'}\pi^{A'} &= f\overline{\eta}_A \\ p^a\sigma_a^{AA'}\overline{\eta}_A &= \overline{f}\pi^{A'}. \end{aligned} \tag{3.13}$$

The second has the form

$$\begin{aligned} p_a\sigma^a_{AA'}\eta^{A'} &= -f\overline{\pi}_A \\ p^a\sigma_a^{AA'}\overline{\pi}_A &= -\overline{f}\eta^{A'}. \end{aligned} \tag{3.14}$$

It is easy to see the difference between (3.13-3.14) and the value $m/\sqrt{2}$ in (3.5-3.6): in (3.13-3.14) we have instead the conjugations of the complex value $f$. One may use polar representation: $f = m\ e^{i\phi}$. In this approach the problem of interpretation of the angle $\phi$ appears. Vaz and Rodrigues [8] had observed an interesting interpretation of the values $m\ cos(\phi)$ and $m\ sin(\phi)$ : these values would be that of longitudinal and transversal masses, respectively.

Let two massless particles be described by two twistors

$$Z^\alpha = (\omega^A, \pi_{A'}), W^\alpha = (\lambda^A, \eta_{A'}), \tag{3.15}$$

respectively. The pair of twistors $(Z^\alpha, W^\alpha)$ represents global Poincaré-covariant coordinates of a point in an eight-dimensional complex symplectic vector space $T \times T$ which defines a phase space for the two massless particles. Coordinates of $(\omega^A, \pi_{B'})$ fulfill its own canonical Poincaré-covariant Poisson bracket algebra introduced in (2.15) and so do the coordinates of $(\lambda^A, \eta_{A'})$.

Assume now that the two massless particles form a massive and, in general, spinning pointlike system. From now on we thus treat $T \times T$ as a reducible phase space of a massive and in general spinning physical system. However, it is necessary to remark, that the two spinors contained in the Dirac's bispinor have different nature, one belongs to a right-handed and the other to a left-handed basis system. In accordance with this identification we identify the linear momentum four-vector of massive particle as

$$p^a = \sigma^a_{AA'}(P_1)^{AA'} + g^{ab}\sigma_b^{AA'}(P_2)_{AA'} = p_1^i + p_2^i, \tag{3.16}$$

where $p_1$ and $p_2$ are momenta belonging to the left- and right- handness massless particles, and

$$(P_1)^{AA'} = \pi^{A'}\overline{\pi}^A, (P_2)_{AA'} = \eta_{A'}\overline{\eta}_A. \tag{3.17}$$

The complex coordinate of the 4-position $Z^{AA'}$ we define as solution of the system

$$\omega^A = iZ^{AA'}\pi_{A'}, \quad \lambda^A = iZ^{AA'}\eta_{A'}, \tag{3.18}$$

Remembering $\pi^A\pi_A = 0$, and $\eta^A\eta_A = 0$, we look for $Z^{AA'}$ in the form

$$Z^{AA'} = \alpha^A\pi^{A'} + \beta^A\eta^{A'}. \tag{3.19}$$

Substitute (3.19) into (3.18) to obtain

$$\omega^A = iZ^{AA'}\pi_{A'} = i\beta^A(\eta^{A'}\pi_{A'}), \lambda^A = iZ^{AA'}\eta_{A'} = i\alpha^A(\pi^{A'}\eta_{A'}), \tag{3.20}$$

use the definition of $f$ given in (3.12), to find

$$\beta^A = -\frac{i}{f}\omega^A, \quad \alpha^A = -\frac{i}{f}\lambda^A. \tag{3.21}$$

This gives the solution of (3.18) as

$$Z^{AA'} = i\frac{1}{f}(\omega^A\eta^{A'} - \lambda^A\pi^{A'}). \tag{3.22}$$

We find the real and imaginary coordinates as real and imaginary parts of $Z^{AA'}$ correspondingly. The real part is given by

$$X^{AA'} = i\frac{1}{2f}(\omega^A\eta^{A'} - \lambda^A\pi^{A'}) - i\frac{1}{2\overline{f}}(\overline{\omega}^{A'}\overline{\eta}^A - \overline{\lambda}^{A'}\overline{\pi}^A). \tag{3.23}$$

For the imaginary part we obtain

$$Y^{AA'} = \frac{1}{2f}(\omega^A\eta^{A'} - \lambda^A\pi^{A'}) + \frac{1}{2\overline{f}}(\overline{\omega}^{A'}\overline{\eta}^A - \overline{\lambda}^{A'}\overline{\pi}^A). \tag{3.24}$$

Using the canonical and covariant twistor coordinates we define the Poincaré-invariant functions according to the following recipe

$$\begin{aligned} e &= Z^\alpha Z_\alpha + W^\alpha W_\alpha = (\omega^A\overline{\pi}_A + \overline{\omega}^{A'}\pi_{A'}) + (\lambda^A\overline{\eta}_A + \overline{\lambda}^{A'}\eta_{A'}),\\ k &= Z^\alpha Z_\alpha - W^\alpha W_\alpha = (\omega^A\overline{\pi}_A + \overline{\omega}^{A'}\pi_{A'}) - (\lambda^A\overline{\eta}_A + \overline{\lambda}^{A'}\eta_{A'}),\\ f &= I_{\alpha\beta}Z^\alpha W^\beta = \pi^{A'}\eta_{A'}, a = Z^\alpha W_\alpha = (\omega^A\overline{\eta}_A + \overline{\lambda}^{A'}\pi_{A'}). \end{aligned} \tag{3.25}$$

We then see the correspondence to the Dirac wave function as the bispinors defined by

$$\begin{aligned} \overline{\Psi}_1 &= \begin{pmatrix} \eta_{A'} & \overline{\pi}^B \end{pmatrix}, \Psi_1 = \begin{pmatrix} \pi^{B'} \\ \overline{\eta}_A \end{pmatrix},\\ \overline{\Psi}_2 &= \begin{pmatrix} \pi_{A'} & \overline{\eta}^A \end{pmatrix}, \Psi_2 = \begin{pmatrix} \eta^{B'} \\ \overline{\pi}_A \end{pmatrix}. \end{aligned} \tag{3.26}$$

From which the MEXOR corresponding to the massive spinning particle is constructed in [7] One may build the basis of four orthonormal vectors given by

$$\begin{aligned} L_1^{AA'} &= \frac{1}{m}(\overline{\pi}^A\pi^{A'} + \overline{\eta}^A\eta^{A'}),\\ L_2^{AA'} &= \frac{i}{m}(\overline{\pi}^A\pi^{A'} - \overline{\eta}^A\eta^{A'}),\\ L_3^{AA'} &= \frac{i}{m}(\overline{\pi}^A\eta^{A'} - \overline{\eta}^A\pi^{A'}),\\ L_4^{AA'} &= \frac{1}{m}(\overline{\pi}^A\eta^{A'} + \overline{\eta}^A\pi^{A'}). \end{aligned} \tag{3.27}$$

And one may expand the vector $Y^{AA'}$ using the basis $\{L_1^{AA'}, L_2^{AA'}, L_3^{AA'}, L_4^{AA'}\}$, one gets

$$Y^{AA'} = l_1L_1^{AA'} + l_2L_2^{AA'} + l_3L_3^{AA'} + l_4L_4^{AA'}, \tag{3.28}$$

the coefficients of the expansion are defined by

$$\begin{aligned} l_1 &= (L_1)_{AA'}Y^{AA'}, \quad l_2 = (L_2)_{AA'}Y^{AA'},\\ l_3 &= (L_3)_{AA'}Y^{AA'}, \quad l_4 = (L_4)_{AA'}Y^{AA'} \end{aligned} \tag{3.29}$$

Substitute $Y^{AA'}$ from (3.24) and taking into account the formulae for $\{e, k, a, \overline{a}\}$ we obtain

$$l_1 = -\frac{e}{2m}, l_2 = -\frac{k}{2m}, l_3 = -\frac{1}{2m}(a - \overline{a}), l_4 = \frac{1}{2m}(a + \overline{a}) \tag{3.30}$$

Introducing (3.27) and (3.30) into (3.29) we get

$$Y^{AA'} = \frac{1}{m^2}(a\overline{\pi}^A\eta^{A'} + \overline{a}\pi^{A'}\overline{\eta}^A - \frac{e+k}{2}\eta^{A'}\overline{\eta}^A - \frac{e-k}{2}\pi^{A'}\overline{\pi}^A). \tag{3.31}$$

Now let us redefine the basis $\{L_1^{AA'}, L_2^{AA'}, L_3^{AA'}, L_4^{AA'}\}$ in vectorial representation via Dirac matrices. For that purpose we introduce the following projective operators

$$\Pi^a = \frac{1}{2m}\gamma_a(1 - i\gamma_5), \quad K^a = \frac{1}{2m}\gamma_a(1 + i\gamma_5). \tag{3.32}$$

Remembering the correspondence given in (3.26) we obtain the mapping $\{L_1^{AA'}, L_2^{AA'}, L_3^{AA'}, L_4^{AA'}\} \to \{l_1^a, l_2^a, l_3^a, l_4^a\}$ from the spinorial representation to vectorial representation

$$\begin{aligned} l_1^a &= \overline{\Psi_1}(\Pi^a + K^a)\Psi_1, \quad l_2^a = i\overline{\Psi_1}(K^a - \Pi^a)\Psi_1, \\ l_3^a &= i\overline{\Psi_1}(\Pi^a - K^a)\Psi_2, \quad l_4^a = \overline{\Psi_1}(\Pi^a + K^a)\Psi_2. \end{aligned} \tag{3.33}$$

In this basis the formulae for vectors $p^a, y^a$ are given by

$$p^a = ml_1^a \text{ and } y^a = \frac{1}{2m}\{(a + \overline{a})l_4^a + i(\overline{a} - a)l_3^a - el_1^a - ikl_2^a\} \tag{3.34}$$

The $T \times T$ is 16 dimensional and up to now we have identified 11+4 variables associated with the massive spinning pointlike system. The sixteenth variable is provided by the angle of rotation of a spacelike two- plane in terms of the orthogonal unit four-vectors. This two-plane polarization may be chosen to be

$$E^{AA'} = \frac{i}{m\sqrt{a\overline{a}}}(a\overline{\pi}^A\eta^{A'} - \overline{a}\pi^{A'}\overline{\eta}^A), \tag{3.35}$$

for $a \neq 0$, and

$$E^{AA'} = \frac{i}{m}(\overline{\pi}^A\eta^{A'} - \pi^{A'}\overline{\eta}^A), \tag{3.36}$$

for $a = 0$. Also we define

$$F^{AA'} = \frac{1}{sm\sqrt{a\overline{a}}}[(k/2)(a\overline{\pi}^A\eta^{A'} + \overline{a}\pi^{A'}\overline{\eta}^A) - a\overline{a}(\pi^{A'}\overline{\pi}^A - \eta^{A'}\overline{\eta}^A)] \tag{3.37}$$

for $a \neq 0$, and

$$F^{AA'} = \frac{i}{m}(\overline{\pi}^A \eta^{A'} + \pi^{A'} \overline{\eta}^A) \tag{3.38}$$

for $a = 0$.

In the vectorial basis these values are written as

$$e^a = \frac{1}{2\sqrt{a\overline{a}}}\{(a+\overline{a})l_3^a + i(\overline{a}-a)l_4^a\},$$

$$for \quad a = 0:$$

$$e^a = l_3^a$$

$$f^a = \frac{1}{2\sqrt{a\overline{a}}}\{(a+\overline{a})l_4^a + i(\overline{a}-a)l_3^a - (a\overline{a})l_3^a\},$$

$$for \quad a = 0:$$

$$f^a = il_4^a \tag{3.39}$$

Using the condition $m^2 = p^a p_a$ and the definition of the orbital angular momentum $M^{ab} = x^a p^b - x^b p^a$ one may construct the relation

$$p_b M^{ab} = x^a m^2 - (p_b x^b) p^a, \tag{3.40}$$

from which one finds

$$x^a m^2 = p_b M^{ab} + (p_b x^b) p^a \tag{3.41}$$

as the coordinates of the position 4-vector. If we substitute the definition of orbital angular momentum into the Pauli—Lubanski four-vector formula (2.4) we obtain $S^a = 0$. This value is non-trivial iff total angular momentum consists besides orbital part also spin part (see, (2.1)). Our goal is to define the spin part by extending the coordinate part. For that purpose let us before consider the massless case. In that case we are expecting that

$$S_a = sp_a. \tag{3.42}$$

To provide this relation we shall look for the structure of the spin part in the following form

$$S^{ab} = L^{abc} p_c = \frac{1}{2} e^{abcd} y_c p_d, \tag{3.43}$$

with

$$L^{abc} = \frac{1}{2} e^{abcd} y_d. \tag{3.44}$$

Substituting this into (2.4) we obtain

$$S_a = -\frac{1}{2} e_{abcd} p^b S^{cd} = -\frac{1}{4} e_{abcd} p^b e^{cdpq} y_p p_q = -p_a (y^b p_b) = s p_a, \tag{3.45}$$

with $(y^b p_b) = -s$.

We suggest that the spin part has the same structure in the massive case, to obtain

$$S_a = -\frac{1}{2} e_{abcd} p^b S^{cd} = -\frac{1}{4} e_{abcd} p^b e^{cdpq} y_p p_q = -p_a (y^b p_b) + m^2 y_a. \tag{3.46}$$

The definition of Pauli—Lubanski four-vector is also based on the suggestion that the total angular momentum is the sum of two operators constructed in analogy with the massless case

$$M_{AA'BB'} = \{ i(\mu_{A'B'} \epsilon_{AB} - \overline{\mu}_{AB} \epsilon_{A'B'}) + i(\nu_{A'B'} \epsilon_{AB} - \overline{\nu}_{AB} \epsilon_{A'B'}), \} \tag{3.47}$$

where

$$\mu_{AB} = \omega_{(A} \overline{\pi}_{B)}, \quad \nu_{AB} = \lambda_{(A} \overline{\eta}_{B)}. \tag{3.48}$$

Let us substitute this formula into the Pauli—Lubanski formula. We write

$$S_a = \frac{1}{2} e_{abcd} P^b M^{cd} = *M_{ab} p^b. \tag{3.49}$$

For the Pauli—Lubanski formula we then obtain

$$\begin{gathered} -S_a = \frac{1}{2} e_{abcd} \{ (p^b_{(1)} + p^b_{(2)})(M^{cd}_{(2)} + M^{cd}_{(1)}) \} = \\ \frac{1}{2}(-s_1 p_{(1)_a} - s_2 p_{(2)_a}) + (1/2) e_{abcd} \{ p^b_{(1)} M^{cd}_{(2)} + p^b_{(2)} M^{cd}_{(1)} \} \end{gathered} \tag{3.50}$$

where $s_1 = \frac{1}{2}(e+k), s_2 = \frac{1}{2}(e-k)$. So, we need to calculate only the expression

$$(1/2) e_{abcd} \{ p^b_{(1)} M^{cd}_{(2)} + p^b_{(2)} M^{cd}_{(1)} \}. \tag{3.51}$$

Using the spinor representations of $p^b and (*M_{cd})$ which are defined by

$$\begin{aligned} *M_{AA'BB'} = \frac{1}{2} \{ & \epsilon_{A'B'} (\omega_A \overline{\pi}_B + \omega_B \overline{\pi}_A) + \epsilon_{AB} (\overline{\omega}_{A'} \pi_{B'} + \overline{\omega}_{B'} \pi_{A'}) + \\ & \epsilon_{A'B'} (\lambda_A \overline{\eta}_B + \lambda_B \overline{\eta}_A) + \epsilon_{AB} (\overline{\lambda}_{A'} \eta_{B'} + \overline{\lambda}_{B'} \eta_{A'}) \} \end{aligned} \tag{3.52}$$

and

$$P^{AA'} = (\pi^{A'}\overline{\pi}^{A} + \eta^{A'}\overline{\eta}^{A}), \tag{3.53}$$

we find

$$*M^{(1)}_{ab}p^{b}_{(2)} = (\omega_A\overline{\pi}_B + \omega_B\overline{\pi}_A)\eta_{A'}\overline{\eta}^{B}) + (\overline{\omega}_{A'}\pi_{B'} + \overline{\omega}_{B'}\pi_{A'})\eta^{B'}\overline{\eta}_A =$$
$$(\overline{\pi}_B\overline{\eta}^{B})\omega_A\eta_{A'} + (\omega_B\overline{\eta}^{B})\eta_{A'}\overline{\pi}_A + (\pi_{B'}\eta^{B'})\overline{\omega}_{A'}\overline{\eta}_A + (\overline{\omega}_{B'}\eta^{B'})\overline{\eta}_A\pi_{A'} \tag{3.54}$$

$$*M^{(2)}_{ab}p^{b}_{(1)} = (\lambda_A\overline{\eta}_B + \lambda_B\overline{\eta}_A)(\pi_{A'}\overline{\pi}^{B} + (\overline{\lambda}_{A'}\eta_{B'} + \overline{\lambda}_{B'}\eta_{A'})(\pi^{B'}\overline{\pi}_A) =$$
$$(\overline{\pi}^{B}\overline{\eta}_B)\lambda_A\pi_{A'} + (\lambda_B\overline{\pi}^{B})\pi_{A'}\overline{\eta}_A + (\pi^{B'}\eta_{B'})\overline{\lambda}_{A'}\overline{\pi}_A + (\overline{\lambda}_{B'}\pi^{B'})\overline{\pi}_A\eta_{A'} \tag{3.55}$$

Collecting all these terms we find

$$-m^2Y_{AA'} = \eta_{A'}\overline{\pi}_A(\omega_B\overline{\eta}^{B} + \overline{\lambda}_{B'}\pi^{B'}) + \overline{\eta}_A\pi_{A'}(\overline{\omega}_{B'}\eta^{B'} + \lambda_B\overline{\pi}^{B}), \tag{3.56}$$

where we have used

$$(\overline{\pi}^{B}\overline{\eta}_B)\lambda_A\pi_{A'} + (\pi^{B'}\eta_{B'})\overline{\lambda}_{A'}\overline{\pi}_A + (\overline{\pi}_B\overline{\eta}^{B}\omega_A\eta_{A'} + (\pi_{B'}\eta^{B'})\overline{\omega}_{A'}\overline{\eta}_A =$$
$$(im^2/2)(Z_{AA'} - \overline{Z}_{AA'}) = -m^2Y_{AA'} \tag{3.57}$$

Substituting all these formulae into (3.50) we obtain

$$\frac{1}{2}\{-\frac{1}{2}(e+k)(\pi_{A'}\overline{\pi}_A) - \frac{1}{2}(e-k)(\eta_{A'}\overline{\eta}_A) - m^2Y_{AA'} +$$
$$\eta_{A'}\overline{\pi}_A(\omega_B\overline{\eta}^{B} + \overline{\lambda}_{B'}\pi^{B'}) + \overline{\eta}_A\pi_{A'}(\overline{\omega}_{B'}\eta^{B'} + \lambda_B\overline{\pi}^{B})\} = \tag{3.58}$$

$$= \frac{1}{2}\{-\frac{1}{2}(e+k)(\pi_{A'}\overline{\pi}_A) - \frac{1}{2}(e-k)(\eta_{A'}\overline{\eta}_A) - m^2Y_{AA'}$$
$$-a\eta_{A'}\overline{\pi}_A - \overline{a}\overline{\eta}_A\pi_{A'}\} = m^2Y^{AA'} + \frac{e}{2}P^{AA'}. \tag{3.59}$$

On the other hand according to (3.46) we can write

$$S^{AA'} = m^2Y^{AA'} + \frac{e}{2}P^{AA'}. \tag{3.60}$$

Indeed the formulae (3.59) and (3.60) are in agreement if we take into account the representation of $Y^{AA'}$ given in (3.31). As a result we obtain following expression for the Pauli-Lubanski vector

$$S^{AA'} = \frac{k}{2}(-\eta^{A'}\overline{\eta}^{A} + \pi^{A'}\overline{\pi}^{A}) + a\overline{\pi}^{A}\eta^{A'} + \overline{a}\pi^{A'}\overline{\eta}^{A}). \tag{3.61}$$

In the vectorial representation this formula is given by

$$s^a = m\{-i\frac{k}{2}l_2^a + (a+\overline{a})l_4^a + i(\overline{a}-a)l_3^a\}, \tag{3.62}$$

in addition we have

$$s^a p_a = 0. \tag{3.63}$$

The vectors $p^a, s^a, e^a, f^a$ form then the set of mutual orthogonal vectors. The spin vector squared is given by

$$s^2 = -(1/m^2)S^a S_a = (1/4)k^2 + a\overline{a}. \tag{3.64}$$

The canonical Poincaré-invariant twistor Poisson algebra on $T \times T$ implies the following physically meaningful Poincaré-covariant commutation relation

$$\{x^a, p^b\} = g^{ab} \tag{3.65}$$

Thus $x^a$ and $p^a$ are conjugate variable but, as we shall see in a moment, they are canonically conjugate if and only if the system is spinless.

Defining the total angular four-momentum

$$L^{ab} := x^a p^b - x^b p^a + \frac{1}{2m^2} e^{abcd} y_c p_d, \tag{3.66}$$

we obtain an important identity

$$\begin{aligned} L^{ab} = i\sigma^a_{AA'}\sigma^b_{BB'}\{(\omega^{(A}\overline{\pi}^{B)}\epsilon^{A'B'} - \overline{\omega}^{A'}\pi^{A')}\epsilon^{AB}) + \\ (\lambda^{(A}\overline{\eta}^{B}\epsilon^{A'B'} - \overline{\lambda}^{A'}\eta^{B')}\epsilon^{AB})\} = M_1^{ab} + M_2^{ab}. \end{aligned} \tag{3.67}$$

In the conclusion of this section let us demonstrate the connection the bispinor representation of momentum with the de Sitter surface equation. Define two vectors with following coordinates

$$\xi^a = \eta^a = (\overline{\pi}^0, \overline{\pi}^1, \pi^{0'}, \pi^{1'}), \eta_a = (\overline{\eta}_0, \overline{\eta}_1, \eta_{0'}, \eta_{1'}). \tag{3.68}$$

Construct the bilinears of $\eta^a, \eta_a$ by using Dirac gamma matrices

$$P_k = \sum_{i,j=1}^{4} \xi^j \{\gamma_k\}^i_j \eta_i, \quad k = 0,1,2,3,4,5. \tag{3.69}$$

For these values given the important following algebraic identity holds

$$P_0^2 - P_1^2 - P_2^2 - P_3^2 + P_4^2 = P_5^2. \tag{3.70}$$

That is the de-Sitter surface equation. Furthermore in conformity with the definition (3.69) one may write also

$$\begin{pmatrix} P_4 & 0 & -P_1 + P_0 & -P_2 + iP_3 \\ 0 & P_4 & P_2 - iP_3 & P_1 + P_0 \\ P_1 + P_0 & P_2 - iP_3 & -P_4 & 0 \\ P_2 + iP_3 & -P_1 + P_0 & 0 & -P_4 \end{pmatrix} \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = $$
$$= P_5 \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}, \tag{3.71}$$

with

$$\begin{aligned} P_0 &= \eta^2\eta_0 + \eta^3\eta_1 + \eta^0\eta_2 + \eta^1\eta_3, \\ P_1 &= -\eta^2\eta_0 + \eta^3\eta_1 + \eta^0\eta_2 - \eta^1\eta_3, \\ P_2 &= -\eta^3\eta_0 - \eta^2\eta_1 + \eta^1\eta_2 + \eta^0\eta_3, \\ P_3 &= i(-\eta^3\eta_0 + \eta^2\eta_1 + \eta^1\eta_2 - \eta^0\eta_3), \\ P_4 &= \eta^0\eta_0 + \eta^1\eta_1 - \eta^2\eta_2 - \eta^3\eta_3, \\ P_5 &= \eta^0\eta_0 + \eta^1\eta_1 + \eta^2\eta_2 + \eta^3\eta_3. \end{aligned} \tag{3.72}$$

Comparing (3.71) with (3.13-3.14) we obtain $P_4 = -2if_2, P_5 = 2f_1$.

## 4. Transformation of the Lorentz-Bargmann-Michel-Telegdi Equations into Oscillator Equations of Motion in the Twistorial Space

We construct the generating function of motion as a simple sum of two Poincaré invariants $m^2$ and $s^2$

$$H(m, s) = m^2 + s^2. \tag{4.1}$$

Taking into account the definitions (3.25) we can write

$$H(m, s) = P^iP_i - \frac{1}{m^2}S^iS_i = 2f\overline{f} + \frac{k^2}{4} + a\overline{a}. \tag{4.2}$$

The essential point is that the values $a, f, k, e$ must be constants of motion, which implies the following form of Hamiltonian function in the variables of twistors

$$\begin{aligned} H(Z^\alpha, W^\beta, Z_\alpha, W_\beta) = \\ \frac{k}{4}(Z^\alpha Z_\alpha + W^\beta W_\beta) + \frac{1}{2} a Z_\alpha W^\alpha + \frac{1}{2}\overline{a} Z^\alpha W_\alpha \\ + f Z_\alpha I^{\alpha\beta} W_\beta + \overline{f} Z^\alpha I_{\alpha\beta} W^\beta \end{aligned} \tag{4.3}$$

Corresponding Hamilton equations for the twistor variables have the form

$$\begin{aligned} i\frac{d}{dt}Z^\alpha = \frac{\partial H}{\partial Z_\alpha}, \quad -i\frac{d}{dt}Z_\alpha = c\frac{\partial H}{\partial Z^\alpha}, \\ i\frac{d}{dt}W^\alpha = \frac{\partial H}{\partial W_\alpha}, \quad -i\frac{d}{dt}W_\alpha = \frac{\partial H}{\partial W^\alpha}, \end{aligned} \tag{4.4}$$

with that choice, the canonical flow is described by the following equations of motion

$$\begin{aligned} i\frac{d}{dt}Z^\alpha - \frac{1}{4}kZ^\alpha - \frac{1}{2}aW^\alpha - fI^{\alpha\beta}W_\beta = 0, \\ i\frac{d}{dt}W^\alpha + \frac{1}{4}kW^\alpha - \frac{1}{2}\overline{a}Z^\alpha + fI^{\alpha\beta}Z_\beta = 0, \end{aligned} \tag{4.5}$$

where $I^{\alpha\beta}$ is the antisymmetric tensor whose unique nonzero components are

$$I_{23} = -I_{32} = 1, \quad I^{01} = -I^{10} = 1.$$

Taking into account the expressions

$$fZ_\alpha I^{\alpha\beta}W_\beta = \overline{\pi}^A\overline{\eta}_A, \quad fZ^\alpha I_{\alpha\beta}W^\beta = \pi^{A'}\eta_{A'}, \tag{4.6}$$

we get the equations of motion

$$\begin{aligned} i\frac{d}{dt}\omega^A - \frac{1}{4}k\omega^A - \frac{1}{2}a\lambda^A + f\overline{\eta}^A = 0, \\ i\frac{d}{dt}\pi_{A'} - \frac{1}{4}k\pi_{A'} - \frac{1}{2}a\eta_{A'} = 0, \\ i\frac{d}{dt}\lambda^A + \frac{1}{4}k\lambda^A - \frac{1}{2}\overline{a}\omega^A + f\overline{\pi}^A = 0, \\ i\frac{d}{dt}\eta_{A'} + \frac{1}{4}k\eta_{A'} - \frac{1}{2}\overline{a}\pi_{A'} = 0. \end{aligned} \tag{4.7}$$

The corresponding complex conjugated equations are

$$\begin{aligned}
-i\frac{d}{dt}\overline{\omega}^{A'} &= \frac{1}{4}k\overline{\omega}^{A'} + \frac{1}{2}\overline{a\lambda}^{A'} - \overline{f}\eta^{A'} = 0,\\
-i\frac{d}{dt}\overline{\pi}_A &= \frac{1}{4}k\overline{\pi}_A + \frac{1}{2}\overline{a\eta_A},\\
-i\frac{d}{dt}\overline{\lambda}^{A'} &= \frac{1}{4}k\overline{\lambda}^{A'} + \frac{1}{2}a\overline{\omega}^{A'} - \overline{f}\pi^{A'} = 0,\\
-i\frac{d}{dt}\overline{\eta}_A &= -\frac{1}{4}k\overline{\eta}_A + \frac{1}{2}a\overline{\pi_A}
\end{aligned} \tag{4.8}$$

Now we shall modify these equations in such a way that $a, f, k, e$ remain constants of motion, choosing the 4-velocity to remain parallel to the 4-momentum, then the resulting equation for the 4-momentum becomes the usual Lorentz force equation. In other words, we require

$$\begin{aligned}
&\frac{d}{dt}(e, f, k, a) = 0,\\
&\frac{d}{dt}X^i = P^i,\\
&\frac{d}{dt}P^i = eF^{ij}P_j.
\end{aligned} \tag{4.9}$$

Following [4] we suggest the next twistor equations in the external electromagnetic field

$$\begin{aligned}
&i\frac{d}{dt}Z^\alpha - \frac{1}{4}kZ^\alpha - \frac{1}{2}aW^\alpha - fI^{\alpha\beta}W_\beta = g^\alpha,\\
&i\frac{d}{dt}W^\alpha + \frac{1}{4}kW^\alpha - \frac{1}{2}\overline{a}Z^\alpha + fI^{\alpha\beta}Z_\beta = h^\alpha.
\end{aligned} \tag{4.10}$$

In coordinate representation we obtain

$$\begin{aligned}
i\frac{d}{dt}\omega^A - \frac{1}{4}k\omega^A - \frac{1}{2}a\lambda^A + f\overline{\eta}^A &= \gamma^A + iX^{AA'}\delta_{A'},\\
i\frac{d}{dt}\pi_{A'} - \frac{1}{4}k\pi_{A'} - \frac{1}{2}a\eta_{A'} &= \delta_{A'},\\
i\frac{d}{dt}\lambda^A + \frac{1}{4}k\lambda^A - \frac{1}{2}\overline{a}\omega^A + f\overline{\pi}^A &= \kappa^A + iX^{AA'}\theta_{A'},\\
i\frac{d}{dt}\eta_{A'} + \frac{1}{4}k\eta_{A'} - \frac{1}{2}\overline{a}\pi_{A'} &= \theta_{A'}.
\end{aligned} \tag{4.11}$$

The corresponding complex conjugated equations with external electromagnetic potential have the form

$$
\begin{aligned}
i\frac{d}{dt}Z_\alpha &= \frac{1}{4}kZ_\alpha + \frac{1}{2}\overline{a}W_\alpha + \overline{f}I_{\alpha\beta}W^\beta + \overline{g}_\alpha,\\
i\frac{d}{dt}W_\alpha &= -\frac{1}{4}kW_\alpha + \frac{1}{2}aZ_\alpha - \overline{f}I^{\alpha\beta}Z^\beta + \overline{h}_\alpha,
\end{aligned}
\tag{4.12}
$$

where $Z_\alpha = (\overline{\pi}_A, \overline{\omega}^{A'}), W_\alpha = (\overline{\eta}_A, \overline{\lambda}^{A'})$. And in coordinate representation are given by

$$
\begin{aligned}
-i\frac{d}{dt}\overline{\omega}^{A'} &= \frac{1}{4}k\overline{\omega}^{A'} + \frac{1}{2}\overline{a}\overline{\lambda}^{A'} - \overline{f}\eta^{A'} + \overline{\gamma}^A - iX^{AA'}\overline{\delta}_A,\\
-i\frac{d}{dt}\overline{\pi}_A &= \frac{1}{4}k\overline{\pi}_A + \frac{1}{2}\overline{a\eta_A} + \overline{\delta}_A,\\
-i\frac{d}{dt}\overline{\lambda}^{A'} &= \frac{1}{4}k\overline{\lambda}^{A'} + \frac{1}{2}a\overline{\omega}^{A'} - \overline{f}\pi^{A'} + \overline{\kappa}^A - iX^{AA'}\overline{\theta}_A,\\
-i\frac{d}{dt}\overline{\eta}_A &= -\frac{1}{4}k\overline{\eta}_A + \frac{1}{2}a\overline{\pi}_A + \overline{\theta}_A.
\end{aligned}
\tag{4.13}
$$

Let us put the external electromagnetic field in spinor notation as follows

$$
F_{AA'BB'} = \frac{i}{e}(\overline{\mu}_{AB}\epsilon_{A'B'} - \mu_{A'B'}\epsilon_{AB}). \tag{4.14}
$$

The functions $g^\alpha, h^\alpha$ in spinorial components are chosen as

$$
g^\alpha = (\gamma^A + iX^{AA'}\delta_{A'}, \delta_{A'}), \quad h^\alpha = (\kappa^A + iX^{AA'}\theta_{A'}, \theta_{A'}), \tag{4.15}
$$

where the spinors $\gamma^A, \delta_{A'}, \kappa^A, \theta_{A'}$ are given along particle's wordline location. Spinors $\gamma^A, \delta_{A'}, \kappa^A, \theta_{A'}$ may also be expressed as

$$
\delta_{A'} = \mu_{A'B'}\xi^{B'}, \theta_{A'} = \mu_{A'B'}\psi^{B'}, \gamma^{A'} = \overline{\mu}^A_B\varsigma^B, \kappa^{A'} = \overline{\mu}^A_B\rho^B, \tag{4.16}
$$

where $\xi^{B'}, \psi^{B'}, \varsigma^B, \rho^B$ are new variable spinors.

Now let us begin our explicit calculations. The condition

$$
\frac{d}{dt}f = 0
$$

yields

$$
\begin{aligned}
\frac{d}{dt}f &= \eta_{A'}\frac{d}{dt}\pi^{A'} + \pi^{A'}\frac{d}{dt}\eta_{A'} =\\
\eta_{A'}(\frac{1}{4}k\pi^{A'} + \frac{1}{2}a\eta^{A'} + \delta^{A'}) &+ \pi^{A'}(-\frac{1}{4}k\eta_{A'} + \frac{1}{2}\overline{a}\pi_{A'} + \theta_{A'}) =\\
&= \eta_{A'}\delta^{A'} + \pi^{A'}\theta_{A'}.
\end{aligned}
\tag{4.17}
$$

Here we used the identity

$$i\overline{\pi}_A f\overline{\eta}^A - i\pi_{A'}\overline{f}\eta^{A'} = 0. \tag{4.18}$$

Inserting $\delta^{A'}, \theta_{A'}$ from (4.16) we get

$$\mu_{A'B'}(\pi^{A'}\psi^{B'} - \eta^{A'}\xi^{B'}) = 0. \tag{4.19}$$

As far as spinors $\psi^{B'}, \xi^{B'}$ are independent of the electromagnetic field this relation is to be fulfilled for arbitrary choice of $\mu_{A'B'}$. Then the following eq. must be satisfied

$$(\pi^{A'}\psi^{B'} - \eta^{A'}\xi^{B'}) = constant\ \epsilon^{A'B'} \tag{4.20}$$

We find two kind of solutions

(1): $\xi^{A'} = y\pi^{A'}, \quad \psi^{A'} = y\eta^{A'},$

(2): $\eta^{A'} = x\pi^{A'}, \quad \psi^{A'} = x\xi^{A'}.$

Now let us check the compatibility of the equations (4.10-4.13) with the Lorentz equation

$$\frac{d}{dt}P^a = eF^{ab}P_b. \tag{4.21}$$

Substituting on the left side $P^a$ by its expression in twistor coordinates one obtains

$$\begin{gathered}\overline{\pi}^A\frac{d}{dt}\pi^{A'} + \pi^{A'}\frac{d}{dt}\overline{\pi}^A + \overline{\eta}^A\frac{d}{dt}\eta^{A'} + \eta^{A'}\frac{d}{dt}\overline{\eta}^A = \\ -i\overline{\pi}^A(\frac{k}{4}\pi^{A'} + \frac{1}{2}a\eta^{A'} + \delta^{A'}) + i\pi^{A'}(\frac{k}{4}\overline{\pi}^A + \frac{1}{2}\overline{a\eta}^A + \overline{\delta}^A) \\ -i\overline{\eta}^A(-\frac{k}{4}\eta^{A'} + \frac{1}{2}\overline{a}\pi^{A'} + \theta^{A'} + i\eta^{A'}(-\frac{k}{4}\overline{\eta}^A + \frac{1}{2}a\overline{\pi}^A + \overline{\theta}^A).\end{gathered} \tag{4.22}$$

Then, for the left side of (4.21) we get

$$i(\pi^{A'}\overline{\delta}^A - \overline{\pi}^A\delta^{A'} + \eta^{A'}\overline{\theta}^A - \overline{\eta}^A\theta^{A'}). \tag{4.23}$$

For the right side of (4.21) write

$$\begin{gathered}eF^{AA'BB'}P_{BB'} = i(-\overline{\mu}_{AB}\epsilon^{A'B'} + \mu_{A'B'}\epsilon^{AB})P_{BB'} = \\ i(-\overline{\mu}_{AB}P_B^{A'} + \mu_{A'B'}P_{B'}^A) = \\ i\{-\overline{\mu}_{AB}(pi^{A'}\overline{\pi}_B + \eta^{A'}\overline{\eta}_B) + \mu_{A'B'}(\pi_{A'}\overline{\pi}_B + \eta_{B'}\overline{\eta}^A)\}.\end{gathered} \tag{4.24}$$

To get

$$
\begin{aligned}
&i\overline{\mu}_{AB}(\pi_{A'}(\overline{\pi}^B-\overline{\xi}^B)+\eta_{A'}(\overline{\eta}^B-\overline{\psi}^B))-\\
&i\mu_{A'B'}(\overline{\pi}_A(\pi^{B'}-\xi^{B'})+\overline{\eta}_A(\eta^{B'}-\psi^{B'}))=0.
\end{aligned}
\tag{4.25}
$$

Let us check with the same procedure the independence of $e$ and $k$ from the evolution parameter

$$
\frac{d}{dt}e=0,\quad \frac{d}{dt}k=0. \tag{4.26}
$$

Write

$$
\begin{aligned}
\frac{d}{dt}(e+k)/2=\overline{\pi}_A\frac{d}{dt}\omega^A+(\frac{d}{dt}\overline{\pi}_A)\omega^A+(\frac{d}{dt}\overline{\omega}^{A'})\pi_{A'}+(\frac{d}{dt}\pi_{A'})\overline{\omega}^{A'}=\\
=-i\overline{\pi}_A(\frac{1}{4}k\omega^A+\frac{1}{2}a\lambda^A-f\overline{\eta}^A+\gamma^A+iX^{AA'}\delta_{A'})\\
+i\omega^A(\frac{1}{4}k\overline{\pi}_A+\frac{1}{2}\overline{a\eta}_A+\overline{\delta}_A)\\
+i\pi_{A'}(\frac{1}{4}k\overline{\omega}^{A'}+\frac{1}{2}\overline{a\lambda}^{A'}-\overline{f}\eta^{A'}+\overline{\gamma}^{A'}-iX^{AA'}\overline{\delta}_A)\\
-i\overline{\omega}^{A'}(\frac{1}{4}k\pi_{A'}+\frac{1}{2}a\eta_{A'}+\delta_{A'})=\\
-ia(\overline{\omega}^{A'}\overline{\eta}_{A'}+\overline{\pi}_A\lambda^A)+i\overline{a}(\omega^A\overline{\eta}_A+\pi_{A'}\overline{\lambda}^{A'})\\
-i\overline{\pi}_A\gamma^A+\overline{\pi}_AX^{AA'}\delta_{A'}+i\omega^A\overline{\delta}_A\\
+i\pi_{A'}\overline{\gamma}^{A'}+\pi_{A'}X^{AA'}\overline{\delta}_A-i\overline{\omega}^{A'}\delta_{A'}.
\end{aligned}
\tag{4.27}
$$

Taking into account the definition $a=(\omega^A\overline{\eta}_A+\overline{\lambda}^{A'}\pi_{A'})$ for the first part we obtain

$$
\begin{aligned}
-i\overline{\pi}_A\gamma^A+\pi_{A'}X^{AA'}\overline{\delta}_A+i\omega^A\overline{\delta}_A\\
=-i\overline{\pi}_A\overline{\mu}^A_C\varsigma^C+\pi_{A'}X^{AA'}\overline{\mu}_{AB}\overline{\pi}^B+i\omega^A\overline{\mu}_{AB}\overline{\pi}^B=\\
i\overline{\mu}_{AB}\overline{\pi}^B(\omega^A-i\pi_{A'}X^{AA'}+\varsigma^A).
\end{aligned}
\tag{4.28}
$$

Remind $\omega^A=i\pi_{A'}(X^{AA'}+iY^{AA'})$ to find

$$
\begin{aligned}
&\overline{\mu}_{AB}\pi^B(Y^{AB'}\pi_{B'}-\varsigma^A)-c.c.=0,\\
&\overline{\mu}_{AB}\eta^B(Y^{AB'}\eta_{B'}-\rho^A)-c.c.=0.
\end{aligned}
\tag{4.29}
$$

Thus, to fulfill (4.25) and (4.29) the following relations should hold

$$\xi_{B'} = \pi_{B'}, \psi_{B'} = \eta_{B'}, \quad \varsigma^A = Y^{AA'}\pi_{A'}, \quad \rho^A = Y^{AA'}\eta_{A'}. \tag{4.30}$$

In the Appendix we explicitly demonstrate that these relations, together with equations for twistors (4.10-4.13), imply the following equations of motion:

$$\frac{d}{dt}S^i = eF^{ij}S_j, \tag{4.31}$$

$$\frac{d}{dt}E^i = -sF^i - eF^{ij}E_j$$
$$\frac{d}{dt}F^i = sE^i - eF^{ij}F_j. \tag{4.32}$$

From (4.31) it follows that the gyromagnetic ratio is necessarily equal to two. Equations (4.32) describe the evolution of the vectors tetrad in the polarization plane.

## Appendix A

Let us deduce the equations

$$\frac{d}{dt}E^i = -sF^i - eF^{ij}E_j,$$
$$\frac{d}{dt}F^i = sE^i - eF^{ij}F_j, \tag{A.1}$$

from twistor equations with the electromagnetic field interaction. Remember the definitions of $E^{AA'}$ and $F^{AA'}$ to compute $\frac{d}{dt}E^{AA'}$.

$$\frac{d}{dt}E^{AA'} = i\frac{i}{m\sqrt{a\overline{a}}}\{a\eta^{A'}(\frac{1}{4}k\overline{\pi}^A + \frac{1}{2}\overline{a\eta}^A + \overline{\delta}^A)$$
$$-a\overline{\pi}^A(-\frac{1}{4}k\eta^{A'} + \frac{1}{2}\overline{a}\pi^{A'} + \theta^{A'}) + \overline{a\eta}^A(\frac{1}{4}k\pi^{A'} + \frac{1}{2}a\eta^{A'} + \delta^{A'})$$
$$-\overline{a}\pi^{A'}(-\frac{1}{4}k\overline{\eta}^A + \frac{1}{2}a\overline{\pi}^A + \overline{\theta}^A)\} \tag{A.2}$$

$$= -\frac{s}{sm\sqrt{a\overline{a}}}\{\frac{k}{4}(a\eta^{A'}\overline{\pi}^A + \overline{a\eta}^A\pi^{A'}) + a\overline{a}(\eta^{A'}\overline{\eta}^A - \pi^{A'}\overline{\pi}^A)\}+$$
$$i\frac{si}{sm\sqrt{a\overline{a}}}\{a(\overline{\mu}^{AB}\eta^{A'}\overline{\pi}_B) + a(-\mu^{A'B'}\eta_{B'}\overline{\pi}^A)+$$
$$\overline{a}(\mu^{A'B'}\pi_{B'}\overline{\eta}^A) - \overline{\mu}^{AB}\overline{\eta}_B\pi^{A'})\}. \tag{A.3}$$

The expression in the first brackets of (A.3) is $(-)sF^{AA'}$, while the expressions in the last brackets (A.3) are as follows

$$e(i/e)(\overline{\mu}^{AB}E^{A'}_B - \mu^{A'B'}E^A_{B'}) = e(i/e)(\overline{\mu}^{AB}\epsilon^{A'B'} - \mu^{A'B'}\epsilon^{AB})E_{BB'} =$$
$$eF^{AA'BB'}E_{BB'}. \tag{A.4}$$

Equating the corresponding terms we obtain

$$\frac{d}{dt}E^i = -sF^i + eF^{ij}E_j.$$

Now let us calculate $\frac{d}{dt}F^{AA'}$.

$$\frac{d}{dt}F^{AA'} = \frac{1}{sm\sqrt{a\overline{a}}}\{\frac{k}{2}\frac{d}{dt}(a\eta^{A'}\overline{\pi}^A + \overline{a}\pi^{A'}\overline{\eta}^A) + a\overline{a}\frac{d}{dt}(\eta^{A'}\overline{\eta}^A - \pi^{A'}\overline{\pi}^A)\} =$$
$$\frac{1}{sm\sqrt{a\overline{a}}}\{\frac{k}{4}(I) + a\overline{a}(II)\}. \tag{A.5}$$

In the bracket {..}: for the part (I)

$$\frac{d}{dt}(a\eta^{A'}\overline{\pi}^A+\overline{a}\pi^{A'}\overline{\eta}^A)=ia\{\eta^{A'}(\frac{k}{4}\overline{\pi}^A+\frac{1}{2}\overline{a\eta}^A+\overline{\delta}^A)$$
$$-\overline{\pi}^A(-\frac{k}{4}\eta^{A'}+\frac{1}{2}\overline{a}\pi^{A'}+\theta^{A'})\}+i\overline{a}\{-\overline{\eta}^A(\frac{k}{4}\pi^{A'}+\frac{1}{2}a\eta^{A'}+\delta^{A'})$$
$$+\pi^{A'}(-\frac{k}{4}\overline{\eta}^A+\frac{1}{2}a\overline{\pi}^A+\overline{\theta}^A)\}=$$
$$=i\{\frac{ka}{2}\eta^{A'}\overline{\pi}^A-\frac{1}{2}\overline{a}a(\overline{\pi}^A\pi^{A'}-\overline{\eta}^A\eta^{A'})\}$$
$$+i\{-\frac{k\overline{a}}{2}\pi^{A'}\overline{\eta}^A+\frac{1}{2}\overline{a}a(\overline{\pi}^A\pi^{A'}-\overline{\eta}^A\eta^{A'})\}+$$
$$ia(\overline{\delta}^A\eta^{A'}-\overline{\pi}^A\theta^{A'})+i\overline{a}(\overline{\theta}^A\pi^{A'}-\overline{\eta}^A\delta^{A'}). \qquad (A.6)$$

For the part (II) we obtain

$$\frac{d}{dt}(\eta^{A'}\overline{\eta}^A-\pi^{A'}\overline{\pi}^A)=i\{\eta^{A'}(-\frac{k}{4}\overline{\eta}^A+\frac{1}{2}a\overline{\pi}^A+\overline{\theta}^A)$$
$$-\overline{\eta}^A(-\frac{k}{4}\eta^{A'}+\frac{1}{2}\overline{a}\pi^{A'}+\theta^{A'})\overline{\pi}^A(\frac{k}{4}\pi^{A'}+\frac{1}{2}a\eta^{A'}+\delta^{A'})$$
$$-\pi^{A'}(\frac{k}{4}\overline{\pi}^A+\frac{1}{2}\overline{a\eta}^A+\overline{\delta}^A)\}=$$
$$i\{(a\eta^{A'}\overline{\pi}^A-\overline{a}\pi^{A'}\overline{\eta}^A)\}+i\{(\overline{\theta}^A\eta^{A'}-\overline{\eta}^A\theta^{A'})+\overline{\pi}^A\delta^{A'}-\pi^{A'}\overline{\delta}^A)\}. \qquad (A.7)$$

Let us examine the parts (I) and (II) with interaction. We obtain for the part (I)

$$ia(\overline{\delta}^A\eta^{A'}-\overline{\pi}^A\theta^{A'})+i\overline{a}(\overline{\theta}^A\pi^{A'}-\overline{\eta}^A\delta^{A'})=$$
$$i[a(\overline{\mu}^{AB}\overline{\pi}_B\eta^{A'}-\mu^{A'B'}\eta_{B'}\overline{\pi}^A)]$$
$$=i[\overline{\mu}^{AB}(a\overline{\pi}_B\eta^{A'}+\overline{a}\pi^{A'}\overline{\eta}_B)-\mu^{A'B'}((a\overline{\pi}^A\eta_{B'}+\overline{a}\pi_{B'}\overline{\eta}^A)]= \qquad (A.8)$$

$$e(i/e)(\overline{\mu}^{AB}\epsilon^{A'B'}-\mu^{A'B'}\epsilon^{AB})(a\overline{\pi}_B\eta_{B'}+\overline{a}\pi_{B'}\overline{\eta}_B), \qquad (A.9)$$

and for the part (II)

$$i\{(\overline{\theta}^A\eta^{A'}-\overline{\eta}^A\theta^{A'})+\overline{\pi}^A\delta^{A'}-\pi^{A'}\overline{\delta}^A)\}= \qquad (A.10)$$

$$i[(\overline{\mu}^{AB}\overline{\eta}_B\eta^{A'}-\overline{\mu}^{AB}\overline{\pi}_B\pi^{A'}+\mu^{A'B'}\pi_{B'}\overline{\pi}^A-\mu^{A'B'}\eta_{B'}\overline{\eta}^A)]=$$
$$=[e(i/e)(\overline{\mu}^{AB}\epsilon^{A'B'}-\mu^{A'B'}\epsilon^{AB})(\overline{\eta}_B\eta_{B'}-\pi_{B'}\overline{\pi}_B). \qquad (A.11)$$

Joining (A.8-A.11) and taking into account the factors $\frac{1}{sm\sqrt{a\overline{a}}}, \frac{k}{2}, a\overline{a}$ we obtain

$$i(\overline{\mu}^{AB}\epsilon^{A'B'} - \mu^{A'B'}\epsilon^{AB})[\frac{1}{sm\sqrt{a\overline{a}}}\{\frac{k}{2}(a\overline{\pi}_B\eta_{B'} + \overline{a}\pi_{B'}\overline{\eta}_B) + a\overline{a}(\overline{\eta}_B\eta_{B'} - \pi_{B'}\overline{\pi}_B)\} = eF^{AA'BB'}F_{BB'}. \tag{A.12}$$

For the part without interaction we have

$$i(k^2/4 + a\overline{a})(a\eta^{A'}\overline{\pi}^A - \overline{a}\pi^{A'}\overline{\eta}^A) = i(s^2)(a\eta^{A'}\overline{\pi}^A - \overline{a}\pi^{A'}\overline{\eta}^A). \tag{A.13}$$

Multiplying by the factor $\frac{1}{sm\sqrt{a\overline{a}}}$ we obtain: $sE^{AA'}$. Substitute the expressions (A.12-A.13) into the right side to get

$$\frac{d}{dt}F^i = sE^i + eF^{ij}F_j.$$

Finally let us check the equation for the spin

$$\frac{d}{dt}S^i = F^{ij}S_j.$$

Where

$$\frac{d}{dt}S^{AA'} = \frac{d}{dt}\{\frac{k}{2}(-\eta^{A'}\overline{\eta}^A + \pi^{A'}\overline{\pi}^A) + a\overline{\pi}^A\eta^{A'} + \overline{a}\pi^{A'}\overline{\eta}^A)\}. \tag{A.14}$$

Again let us calculate separately two terms in the bracket {..}: For the part (I)

$$\begin{aligned}\frac{d}{dt}(a\eta^{A'}\overline{\pi}^A + \overline{a}\pi^{A'}\overline{\eta}^A) = \\ ia\{\eta^{A'}(\frac{k}{4}\overline{\pi}^A + \frac{1}{2}\overline{a}\overline{\eta}^A + \overline{\delta}^A) - \overline{\pi}^A(-\frac{k}{4}\eta^{A'} + \frac{1}{2}\overline{a}\pi^{A'} + \theta^{A'})\} \\ +i\overline{a}\{-\overline{\eta}^A(\frac{k}{4}\pi^{A'} + \frac{1}{2}a\eta^{A'} + \delta^{A'}) + \pi^{A'}(-\frac{k}{4}\overline{\eta}^A + \frac{1}{2}a\overline{\pi}^A + \overline{\theta}^A)\} =\end{aligned} \tag{A.15}$$

$$\begin{aligned}= i\{\frac{ka}{2}\eta^{A'}\overline{\pi}^A - \frac{1}{2}\overline{a}a(\overline{\pi}^A\pi^{A'} - \overline{\eta}^A\eta^{A'})\} \\ +i\{-\frac{k\overline{a}}{2}\pi^{A'}\overline{\eta}^A + \frac{1}{2}\overline{a}a(\overline{\pi}^A\pi^{A'} - \overline{\eta}^A\eta^{A'})\} + \\ ia(\overline{\delta}^A\eta^{A'} - \overline{\pi}^A\theta^{A'}) + i\overline{a}(\overline{\theta}^A\pi^{A'} - \overline{\eta}^A\delta^{A'}).\end{aligned} \tag{A.16}$$

For the part (II)

$$\frac{d}{dt}(\eta^{A'}\overline{\eta}^{A}-\pi^{A'}\overline{\pi}^{A})=$$
$$i\{\eta^{A'}(-\frac{k}{4}\overline{\eta}^{A}+\frac{1}{2}a\overline{\pi}^{A}+\overline{\theta}^{A})-\overline{\eta}^{A}(-\frac{k}{4}\eta^{A'}+\frac{1}{2}\overline{a}\pi^{A'}+\theta^{A'})$$
$$\overline{\pi}^{A}(\frac{k}{4}\pi^{A'}+\frac{1}{2}a\eta^{A'}+\delta^{A'})-\pi^{A'}(\frac{k}{4}\overline{\pi}^{A}+\frac{1}{2}\overline{a\eta}^{A}+\overline{\delta}^{A})\}=$$
$$i\{(a\eta^{A'}\overline{\pi}^{A}-\overline{a}\pi^{A'}\overline{\eta}^{A})\}+i\{(\overline{\theta}^{A}\eta^{A'}-\overline{\eta}^{A}\theta^{A'})+\overline{\pi}^{A}\delta^{A'}-\pi^{A'}\overline{\delta}^{A})\}. \quad (A.17)$$

Now let us examine the parts (I) and (II) with interaction. We obtain for the part (I)

$$ia(\overline{\delta}^{A}\eta^{A'}-\overline{\pi}^{A}\theta^{A'})+i\overline{a}(\overline{\theta}^{A}\pi^{A'}-\overline{\eta}^{A}\delta^{A'})=$$
$$i[a(\overline{\mu}^{AB}\overline{\pi}_{B}\eta^{A'}-\mu^{A'B'}\eta_{B'}\overline{\pi}^{A})]$$
$$=i[\overline{\mu}^{AB}(a\overline{\pi}_{B}\eta^{A'}+\overline{a}\pi^{A'}\overline{\eta}_{B})-\mu^{A'B'}((a\overline{\pi}^{A}\eta_{B'}+\overline{a}\pi_{B'}\overline{\eta}^{A})]=$$
$$i(\overline{\mu}^{AB}\epsilon^{A'B'}-\mu^{A'B'}\epsilon^{AB})(a\overline{\pi}_{B}\eta_{B'}+\overline{a}\pi_{B'}\overline{\eta}_{B}). \quad (A.18)$$

and for the part (II)

$$i\{(\overline{\theta}^{A}\eta^{A'}-\overline{\eta}^{A}\theta^{A'})+\overline{\pi}^{A}\delta^{A'}-\pi^{A'}\overline{\delta}^{A})\}= \quad (A.19)$$

$$i[(\overline{\mu}^{AB}\overline{\eta}_{B}\eta^{A'}-\overline{\mu}^{AB}\overline{\pi}_{B}\pi^{A'}+\mu^{A'B'}\pi_{B'}\overline{\pi}^{A}-\mu^{A'B'}\eta_{B'}\overline{\eta}^{A})]=$$
$$=[i(\overline{\mu}^{AB}\epsilon^{A'B'}-\mu^{A'B'}\epsilon^{AB})(\overline{\eta}_{B}\eta_{B'}-\pi_{B'}\overline{\pi}_{B}). \quad (A.20)$$

Joining (A.17-A.20) and taking into account the factor $\frac{k}{2}$, we obtain

$$i(\overline{\mu}^{AB}\epsilon^{A'B'}-\mu^{A'B'}\epsilon^{AB})[\frac{k}{2}(-\eta_{B'}\overline{\eta}_{B}+\pi_{B'}\overline{\pi}^{B})+(a\overline{\pi}_{B}\eta_{B'}+\overline{a}\pi_{B'}\overline{\eta}_{B})]=$$
$$eF^{AA'BB'}S_{BB'}. \quad (A.21)$$

It gives the expected relation

$$\frac{d}{dt}S^{i}=eF^{ij}S_{j}.$$

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