THE GEOMETRIC CONTENT OF THE ELECTRON THEORY. (PART II) THEORY OF THE ELECTRON FROM START

Jaime Keller
División de Estudios de Posgrado, Facultad de Química, and
Facultad de Estudios Superiores-Cuautitlán,
Universidad Nacional Autónoma de México,
Apartado 70-528, 05410 México, D.F., MEXICO, (permanent address) and
Federal Institute of Technology (ETH), Festkoerperphysik
ETH-Zuerich, CH-8093, Switzerland.
e-mail keller@servidor.unam.mx and keller@cms.tuwien.ac.at

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To Prof. H. C. Siegmann on his 65th. birthday.

Abstract. From the considerations of the previous paper on the geometric content of the electron theory and the basic principles of the space-time-action relativity theory (START) we formulate a comprehensive and complete theory of the electron. Our approach contains, being a deductive theory, the results of density functional theory, wave function quantum mechanics, the classical theory of the electron, the description of the electron as a lepton in elementary particles theory and the fundamentals of both electrodynamics and electroweak interactions. The approach is otherwise selfcontained.

Contents

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1. Introduction

In the first part of this series (The Geometric Content of the Electron Theory Part I [72], here **ET1**) we have reviewed the geometric content of both the classical and the quantum theory of the electron. There we analyzed the duplicity of the description of energy content in the classical approach to the theory of the electron, typically the total mass being expressed both as the electromagnetic energy of the electron fields and the mass content of the electron as a particle, also it was very important to point out that considering the charges distributions of an atomic system (hydrogen was the obvious example) the change in energies from separated particles to the atomic case, with the density distribution description given as from quantum mechanics, showed that the "atomic energy" corresponded to the change in the energies of the electromagnetic fields of the nucleus and the electron. In the analysis of the wave mechanics approach to the theory of the electron we also met with some puzzling double descriptions both at the level of the quantum mechanics formulation and, for example, facts like the spin of the electron being contained both in the electron field and in the electromagnetic field of the electron. The multivector approach to the analysis of the electron quantum mechanics was also included.

In **ET1** we mention that it was generally accepted that a more fundamental approach to the theory of the electron seems to be required. In the present paper we will develop such an approach, from first principles, from a very basic geometric formulation of the physical nature, using as an starting point the concept of the physical world being described as an energy distribution over space, which from a relativistic point of view corresponds to an action distribution over spacetime. We have developed this approach in a series of papers for the analysis of general relativity, quantum mechanics and of the theory of elementary particles. Here we will use it as a basic frame of reference to construct a comprehensive theory of the electron. We will develop this theory to cover most aspects of the theory of the electron including those related to the theory of elementary particles and to general relativity. By necessity we will have to consider basic problems in quantum electrodynamics, gauge theory and of the theory of induced mass and charges.

In section 2 we will present the space-time-action relativity theory (START) in a form suitable for our present purposes, in section 3 we will present a formulation of density functional theory and of wave quantum mechanics, as derived from START,and a general formulation of a theory of interacting elementary particles. Finally in section 4 we present a theory of the electron. In the appendices we provide the necessary mathematical formulations, as used in the text, and some historical and mathematical remarks about the theory of the electron.

2. Space-Time-Action Relativity Theory

In a series of papers [52]-[78] we have presented and developed a classical theory of fields in (complex) spacetime geometry and arrived to the conclusion that this geometry corresponds to a unified space-time-action geometry. We started from three basic assumptions: a) The use of spacetime as a basic frame of reference, b) The introduction of physical phenomena as an action-density over spacetime and c) The geometrical (vectorial) union of space, time and action in a quadratic space where a relativistic condition $(dS)^2 = 0$ defines both kinematics and dynamics. The basic principles of this Space-Time-Action Relativity Theory (START) were presented and related to our present knowledge of the basic structures of physics.

2.1. Motivation for the Use of Space-Time-Action Geometry

For the construction of the vector representation of the space-time-action geometry. We should start by first considering some properties of the action dimension.

In particular, for the analysis of the theory of the electron we must be reminded that there exists a numerical and dimensional relationship between the fundamental constants:

$$g_D e = (\frac{\hbar}{2})c = \frac{hc}{4\pi} \quad ; \quad \boldsymbol{\mu}_e = g_D r_0 = e r_{compton} = e\mathbf{s}/m_0 \quad , \tag{1}$$

relating the Dirac monopole charge g_D, the electron's electrical charge e, the unit of action h and the numerical value of the velocity of light c, $\boldsymbol{\mu}_e$ the magnetic moment of the electron, $r_{compton}$ the Compton wavelength, r_0 the classical electron radius, m_0 the electron mass and $\mathbf{s}$ the electron spin $|\mathbf{s}| = \hbar/2$.

In spacetime geometry, spanned by the vectors $\{e_\mu;\ \mu = 0, 1, 2, 3\}$ of the spacetime geometry G_{ST}, the constants e and c are invariant scalar quantities. That is they are invariant under proper or improper Lorentz transformations. In the basic relations of electromagnetism a moving charge would appear as a four current

$$\mathbf{j} = j^\mu e_\mu \quad , \tag{2}$$

where $j^\mu = ev^\mu$ with v^μ the four velocity. The Maxwell equations would then relate $\mathbf{j}$ to a vector four potential

$$\partial^\mu \partial_\mu \mathbf{j} = \mathbf{A} \quad , \tag{3}$$

and to the (bivector) Electromagnetic Strengths

$$\mathbf{F} = F^{\mu\nu} e_\mu e_\nu = \frac{1}{2}(\partial^\mu A^\nu) e_\mu e_\nu \quad . \tag{4}$$

On the other hand, if a magnetic monopole $g_{_D}$ existed it will, when moving, originate a **trivector** current (usually called axial vector four current) $\mathbf{k}$

$$\mathbf{k} = \mathbf{g}_{_D} \mathbf{v} = \mathbf{g}_{_D} v^\mu e_\mu = k^{\lambda\nu\rho} e_\lambda e_\nu e_\rho \quad , \tag{5}$$

which will, through the natural extensions of the Maxwell equations to magnetic currents, originate an axial vector potential

$$\partial^\mu \partial_\mu \mathbf{k} = \mathbf{B} = b^{\nu\lambda\rho} e_\nu e_\lambda e_\rho \quad , \tag{6}$$

where all indexes μ, ν, λ, ρ are different among themselves. Then, geometrically, the "constant" $\mathbf{g}_{_D}$ must have a (real or imaginary) pseudoscalar property, that is $\mathbf{g}_{_D} = g_{_D} e_5$ which results in $\mathbf{k}$ and $\mathbf{B}$ being axial vectors and bivector Electromagnetic Fields Strengths

$$e_\mu \partial^\mu \mathbf{B} = \mathbf{F} = \frac{1}{3}(\partial_\mu B^{\mu\lambda\rho} - \partial_\mu B^{\lambda\mu\rho} + \partial_\mu B^{\lambda\rho\mu}) e_\lambda e_\rho \quad . \tag{7}$$

Here, in (5), (6) and (7), we have used the multivector scalar product property $\frac{1}{2}(e_\mu e_\nu + e_\nu e_\mu) = g_{\mu\nu}\mathbf{1}$ with $\mathbf{1}$ the scalar unit of the geometric algebra and $g_{\mu\nu}$ the metric tensor. This is the reason to have used the symbol $\mathbf{g}_{_D} = g_{_D} e_5$ in equation (5) above, that is the magnetic monopole strength multiplied by the unit pseudoscalar e_5.

Then geometrically in (1) there are two possibilities, either there is some factor missing or action has also Spacetime Geometry (real or imaginary) pseudoscalar properties. We should remember that spin and the Compton radius are axial vector quantities. The analysis below will show that, if considered jointly with spacetime, this second option is the more useful identification of the geometric properties of the action. Otherwise, when action is considered separately, it is always properly represented as a scalar quantity. The action constant is then, geometrically, both the constant relating energy-momentum to spacetime and the key to the construction of an unified geometry of space, time and action. The new geometry is derived from the introduction of an action coordinate $x^5 = a(\mathbf{x})\frac{d_0}{h}$ where a is the density of action at a given point

$\mathbf{x}$ of spacetime, h Planck's constant and d_0 an invariant basic length to be determined below, basically $x^5 = \kappa_0 a(\mathbf{x})$. We use the traditional indexes 0,1,2,3 for time and space and, below, the isomorphism between the Dirac gamma symbols with the vectors in the geometry of spacetime $e_\mu \Rightarrow \gamma_\mu$ and, when necessary the representation of the γ_μ by complex 4×4 matrices.

In fact the special property of the pseudoscalar in spacetime $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ (or in the notation above $e_5 = e_0e_1e_2e_3$) is that $\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$ (from $\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu$, $\mu \neq \nu$) and then it has the same commuting properties with the generating vectors than the generating vectors among themselves. We have discussed elsewhere [55, 62] that the linearly independent multivector $i\gamma_5$ is then the immediate candidate to introduce an additional basis vector, adding one more dimension and, through its use, obtain the five dimensional carrier space spanned by the basic vectors e_v, $v = 0, 1, 2, 3, 5$ (identified as $e_0 = \gamma_0$, $e_1 = \gamma_1$, $e_2 = \gamma_2$, $e_3 = \gamma_3$ and $e_5 = i\gamma_5$) with metric $g_{uv} = diag(1, -1, -1, -1, 1)$. Its use allows the construction of a geometrical frame of work for the description of physical processes: a unified space-time-action geometry G_{STA}.

The product of $e_5 = i\gamma_5$ with any element of the original Spacetime Geometry G_{ST} is in the basis vector set a purely imaginary quantity with the result that the $2(2^4) = 2^5 = 32$ elements of the new (space-time-action) geometry are equivalent to $G_{STA} = G_{ST} \otimes C$. In the G_{STA} geometry the coordinates are real numbers.

The value of d_0 we will use below (taken from the theory of the electron) will be the axial character $d_0 = r_{compton} = r_0/2\alpha$, the Compton radius, where $r_0 = e^2/m_0c^2$, the radius that relates the mass of the electron to an electromagnetic equivalent energy and also $r_0 = \boldsymbol{\mu}/g_{_D}c$ the ratio of the electron magnetic moment to the Dirac monopole magnetic charge. With this choice the presentation of the theory will immediately be suitable for the study of elementary particles. Nevertheless we will show that the same units are practical in the study of gravitational interactions.

2.2. Spacetime to Space-Time–Action

Our approach in the past has been an inductive process, searching for the different geometrical structures in fundamental physics. In recent papers [78, 79] we have shown that once we know this structures and their geometry, a deductive presentation is possible and useful.

2.2.1. Formal Presentation

The ideas developed in the space-time-action relativity theory (START) can be derived from the systematic use of the following principles and postulates.

First Principle: Constancy of the **velocity** of light in vacuum. Independently of the state of movement of the source or of the observer (Einstein Relativity).

First Postulate: There is a geometry, corresponding to spacetime, where Principle 1 is fulfilled (Minkowski space-time with local pseudo euclidean structure).

Second Principle: Constancy of the **action** of light as an elementary physical phenomenon independent of the state of movement of the source or of the observer. (Below this principle will be reformulated as: Elementary physical phenomena correspond to a density of processes moving at the speed of light in the unified Space-Time-Action geometry).

Second Postulate: There is a geometry, corresponding to the union of spacetime and action where Principles 1 and 2 are fulfilled (pseudoeuclidean structure).

Main Theorem: KT. The geometry where Postulate 2 is fulfilled is a five dimensional basis geometry, mathematically corresponding to a particular complexification of spacetime.

Third Principle: The changes in action always occur as integer multiples of h. Equivalent to the action per cycle being an integer multiple of $\hbar/2$. (This has to be a constitutive part of the units and practical use of **KT** theorem).

The relation between a 5-dimensional geometry and the complexification of the basis set has been presented in the introduction and will be thoroughly discussed below.

Proof of KT: We have the kinematical concept of trajectory ($\mu,\nu = 0,1,2,3$)

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad , \tag{8}$$

generated by the dx^μ and we want to include as a fifth coordinate the dynamical concept of action

$$dA = p_\mu dx^\mu = g_{\mu\nu} p^\mu dx^\nu \quad , \tag{9}$$

which defines $p_\mu = \partial A/\partial x^\mu$, then write $p_\mu = \tan\mathbf{\Theta}(\boldsymbol{\mu})$ and join formally, defining $-\mathbf{i^2} = \mathbf{1}$, $(\frac{s_0}{h})^\mathbf{2} = \boldsymbol{\kappa}_0^2$

$$dS^\mu = dx^\mu(\mathbf{1} + \mathbf{i}\boldsymbol{\kappa}_0 \tan\mathbf{\Theta}(\boldsymbol{\mu})) \quad , \tag{10}$$

to obtain from the real quadratic form

$$dS^2 = g_{\mu\nu}\frac{1}{2}\left\{dx^\mu(\mathbf{1} + \mathbf{i}\boldsymbol{\kappa}_0 \tan\mathbf{\Theta}(\mu))\ dx^\nu(\mathbf{1} - \mathbf{i}\boldsymbol{\kappa}_\mathbf{0} \tan\mathbf{\Theta}(\boldsymbol{\nu})) + \mathbf{c.c.}\right\} , \tag{11}$$

and, finally

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu \left(\mathbf{1} + \boldsymbol{\kappa}_{\mathbf{0}}^{\mathbf{2}} \tan \boldsymbol{\Theta}(\boldsymbol{\mu}) \tan \boldsymbol{\Theta}(\boldsymbol{\nu}) \right) \quad , \tag{12}$$

or, in five dimensional formulation

$$dS^2 = g_{uv} dx^u dx^v = ds^2 + \boldsymbol{\kappa}_0^2 dA^2 \, ; \, u, v = 0, 1, 2, 3, 5 \quad , \tag{13}$$

where $\boldsymbol{\kappa}_0^2 dA^2$ corresponds to the square of a density of action. The basic dynamical equation is

$$\delta \int dS^2 = 0 \quad , \tag{14}$$

in a joint minimization of trajectory and action. Gravitation requires shortest trajectories and the common procedure of Lagrangian minimization, the minimization of action. The universal constant $\boldsymbol{\kappa}_0$ expresses the action as an equivalent distance and $(dx^5)^2 = (\boldsymbol{\kappa}_0 dA)^2$, with $g_{uv} = diag(1, -1, -1, -1, 1)$ defines the metric of the equivalent five dimensional geometry basis vectors. **.- end proof of KT.**

This is a relativistic approach we have called START.

For the study of the units to be used in this unified geometry consider the definition

$$m_0 c^2 = g_D \frac{e}{r_{\text{compton}}}, \, r_{\text{compton}} = \frac{\hbar}{2 m_0 c} = \frac{r_0}{2\alpha}, \, \alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137} \quad , \tag{15}$$

showing the relation between mass and $\boldsymbol{\kappa}_0 = d_0/h = 4\pi r_{\text{compton}}/h = 1/m_0 c$, that is a unit curvature in the action direction would correspond to the energy of the electron mass. The classical limit of the unification of action to spacetime is obtained when $\boldsymbol{\kappa}_0 \to \infty$ in a form similar to the classical limit of the unification of space and time being obtained when $c \to \infty$. Note that $\kappa_0 >> c$. Space-time-action and the unification of minimal trajectories and minimal action will be relevant mainly in the field of high energy physics.

This set of principles and postulates, together with the minimum action principle $\delta(dS^2) = 0$, defines the START theory. In this approach the physical world corresponds to a 4-dimensional surface in the space-time-action geometry. Elementary phenomena to exchange of multiples of the elementary action h between distributions of action provided that this exchanges preserve the local symmetry constrains. A creation of an elementary particles pair of fields (distributions) requires that the sum of symmetry properties cancels except for pre-existing constrains in the action distribution which results in the new fields. Action is conserved, then locally in time energy-momentum is conserved but in short times and distances only the overall conservation rules and local fluctuations are not excluded by the formalism. This will be the guiding principles below.

2.3. The Representation of the START Geometry

From the considerations above the fifth coordinate corresponds to **action density**, not to the accumulated action. Because action for a given amount of energy distribution in space would be a ever increasing function of time, even if local in time changes are meaningful,a natural representation could also be the use of a **rate of change of action** over spacetime. This will be a moving in time distribution of energy in space. A representation closer to our common visualization of nature. In this representation large enough densities of energy at rest with respect to an observer will be seen as mass distributions. A more mathematical, less intuitive, representation would be that of a circular complex function of the action $a(\mathbf{X})$ of the form

$$\boldsymbol{\Psi}\gamma^0\boldsymbol{\Psi}^\dagger = \boldsymbol{\rho} e^{-\mathbf{i}a(X)} \quad , \tag{16}$$

where $\boldsymbol{\rho} = \boldsymbol{\rho}(X)$ is a carrier density and $\mathbf{i}$ would be a proper multivector such that $\mathbf{i}^2 = -\mathbf{1}$.

The representation in (16) of the action density leads immediately to the concept of an auxiliary wave function $\boldsymbol{\Psi}$ and to the possibility of selecting the multivector imaginary unit in relation to our third principle above as far as the generators of angular momentum are the (bivectors in STA) $i\gamma_{ij}$. Angular momentum is expressed in the same units as action see (1) above. Also $\boldsymbol{\Psi}$ has to be a multivector valued function of the spacetime coordinates $\mathbf{X}$. The energy momentum operator $\widehat{\mathbf{p}}$ would then be

$$\widehat{\mathbf{p}} = \mathbf{i}\hbar\gamma^\mu\partial_\mu \quad . \tag{17}$$

The inclusion of γ^0 in (16) is needed to specify the observer's frame of reference to which the energy distribution is related. For that "observer", as a reference point, time is the only evolving variable.

With this presentation we recover the formulation of the quantum mechanics of the electron in terms of multivectors as presented in appendix B.

3. Physical Structures

We should now consider the physical structures related to matter and its interaction fields.

3.1. Action Density Functional Theory and Wave Function Theory

In the space defined above we now introduce a dimensionless function $\mathbf{K} = \mathbf{K}(\mathbf{X})$ defined at each point $\mathbf{X}$ of spacetime.

Here we have to make two crucial considerations about what we know about action and about quantum mechanics that will be basic for the systematic procedure presented in this section.

As we mention in the previous section for each observer the concept of action is locally the study of the energy attributed to physical phenomena. Relativity theory showed that an energy-momentum content in a given volume presents an inertia M, mass, through the basic relation $M = E/c^2$, this total energy content is independent of the form we have chosen to describe that matter. Let us for example start by considering that we have a free particle of a solid material. Which means that we have chosen a macroscopic point of view and a separation into the material itself and some external forces representing the mutual interaction between the material and the rest of the physical system. For some practical purposes this could be a sufficient degree of description where only shape and density will be required. We also know that if we consider that this piece of material is in movement relative to some measuring device, quantum mechanics can be applied to the material as a unit. Otherwise we may consider that the solid particle consists of molecules in interaction, and that there is an effective interaction potential between the molecules (this is a very common case in the study, for example, of rare gas solids). Quantum mechanics should be applied again for this system of interacting molecules. If our decision is to describe the material as electrons and nuclei we will again apply quantum mechanics at this level of detail in the description. The next steps, the study of the nucleus or the study of the nucleons are again admissible. In every case we will have a total energy which should be equal to the total energy of the previous steps and we will have the practical choice of separation at any degree of description into: constitutional energy or mass, kinetic energy and interaction energy.

That is: quantum mechanics is a universal description of the phenomena, valid for any degree of detail we might have chosen for the description and can not be a property of the components but a basic property for the description of nature. Action and spacetime are fundamental concepts in the description of nature and not concepts dependent on the system we are describing.

But, because particle density and density of action are gauge invariant physical quantities, we need to develop a procedure which can allow gauge freedom, that is allow for arbitrary but correct and useful descriptions. This is possible with the introduction of the probability amplitude known as wave function ψ, required to contain the necessary information in a form compatible to the basic concept that the energy-momentum components are obtained by using the operator $i\hbar\partial_\mu$ applied to the function which describes the splitting of the action density into a particle density ρ and the action per particle. The definition $\rho = \psi^2$ allows gauge independence. This procedure can be carried at any level of description, hence the universality of the possibilities to use Wave Equations in Quantum Mechanics.

Once we have established that we are: 1.- defining a density of action in spacetime which corresponds to an energy density in space $E(\mathbf{x}) = \mathbf{i}\hbar\partial\mathbf{K}(\mathbf{X})/\partial t$ for a fixed observer, for the reason to use $\mathbf{i}$ see below in this section, and 2.- the universality of the description, which allows a choice of the level of detail (for example: molecules→ atoms→ electron and nuclei to nucleons → quarks, provided that at each step the decision is formally made by selection of the type of "particle" and by the type of interaction between particle and the internal energy of the particles) we can now proceed to the steps creating a practical density functional theory:

The energy density is written as a product of a particle density and a (global) energy per particle.

$$\begin{aligned} E &= \int E(\mathbf{x})d\mathbf{v} = \int \rho(\mathbf{x})\varepsilon d\mathbf{v} \\ &= \int \rho(\mathbf{x})(\mathbf{m}_c c^2 + \mathbf{kin}(\mathbf{x}) + \mathbf{V}(\mathbf{x}) + \mathbf{V}_{xc}(\mathbf{x}) + \varepsilon_0(\mathbf{x}))d\mathbf{v} \quad , \end{aligned} \tag{18}$$

where we have defined the energy density $E(\mathbf{x})$, the particle density $\rho(\mathbf{x})$, the, by definition of carriers, actual kinetic energy per particle $\mathbf{kin}(\mathbf{x})$, the external and average internal potential energy per particle $\mathbf{V}(\mathbf{x})$, the correction to the average kinetic and potential energy per particle arising from the statistics of the type of particles under consideration $\mathbf{V}_{xc}(\mathbf{x})$, and a local energy $\varepsilon_0(\mathbf{x})$, basic term required to compensate for any difference in the sum of the previous terms with respect to the average energy per particle ε. **Density functional theory** describes the self organization of the system with density $\rho(\mathbf{x})$.

In (18) we are in fact defining the carriers. First when we consider the energy density to be given by the product $\rho(\mathbf{x})\varepsilon$ of a density of carriers and an average energy per carrier, the same for all, in a form which make them indistinguishable. In that integral the domain of integration defines the system of carriers, within this domain all are equivalent. In the last term, when we refer to the kinetic energy, we consider these carriers to be independent particles, that is the reason for the last function to be required to compensate both for the definition of carriers and for considering them as independent particles.

When we enter into the theory of the electron we will consider that within our START formalism there are some elementary carriers (they will be identified as either massless fields or the electron field), all other carriers will corresponds to less fundamental descriptions and then will require either the last terms in (18) or a procedure, described below, of introducing gauge fields and the action associated with them.

3.2. The Density as the Basic Variable

It is convenient to define the action in a form that distinguishes the part corresponding to the self-organization of the distribution and the part which corresponds to the "external" influences on the distribution. If the external influence is represented by the external potential $\mathbf{v}(\mathbf{X})$ we can write for the total action

$$A = \int dt\, (E_I[\rho(\mathbf{X})] + \int d\mathbf{X}\, \mathbf{v}(\mathbf{X})\rho(\mathbf{X})) \quad , \tag{19}$$

where the functional $E_I[\rho(\mathbf{X})]$ corresponds to the energy of the distribution of carriers $\rho(\mathbf{X})$. This functional E_I has the interesting property

$$\frac{\delta E_I}{\delta \rho(\mathbf{X})} = -\mathbf{v}(\mathbf{X}) \quad . \tag{20}$$

This is a basic relation in action-DFT as far as there is an intrinsic definition of the external potential. This shows the tautological nature of the concept of carriers, once they are defined the external potential is defined through the definition of the carriers themselves by $E_I[\rho(\mathbf{X})]$. The tautological cycle is closed when given $\mathbf{v}(\mathbf{X})$ and $\rho(\mathbf{X})$ the kinetic energy and the interaction terms define $E_I[\rho(\mathbf{X})]$.

3.3. Gauge Freedom for the Description of the Energy

The fact that we are arbitrarily defining the terms above requires the possibility of changing the description of the energy partitioning without changing the description of the density. That is that the density $\rho(\mathbf{x})$ is required to be gauge invariant whereas the description of the energy (action) is gauge dependant. This is achieved by constructing the energy density as the product of two conjugated quantities $\Psi(\mathbf{x})$ and $\Psi^\dagger(\mathbf{x})$ such that $\rho(\mathbf{x}) = \Psi^\dagger(\mathbf{x})\Psi(\mathbf{x})$ is gauge invariant. Here we have defined an auxiliary quantity which can be essentially written in terms of the basic action $a(\mathbf{x})$ and the action introduced by the gauge freedom $\phi(\mathbf{x})$ in units of $\hbar$, as

$$\Psi(\mathbf{x}) = \sqrt{\rho(\mathbf{x})}\; e^{-ia(\mathbf{x})+i\phi(\mathbf{x})} \quad , \tag{21}$$

where we are restricted, by definition, to

$$\hbar\partial(a(\mathbf{x})+\phi(\mathbf{x}))/\partial t=\boldsymbol{\varepsilon} \quad , \tag{22}$$

showing the gauge freedom of the description of the energy associated with the particle. In a sense at all position points $\mathbf{x}$ we have the same energy per particle $\boldsymbol{\varepsilon}$ which only in the simplest cases would be the sum of a kinetic and a potential energy part in the traditional sense. A well known example is the case of electron density functional theory where the kinetic energy is assumed the kinetic energy of the free electron gas and then our term $\varepsilon_0(\mathbf{x})$ will contain, among other terms the difference between the actual kinetic energy and the free electron gas term. The term $\varepsilon_0(\mathbf{x})$ exists either from the incomplete description of the other terms, the usual case, or from inaccuracies in the computational procedure. It acts locally to distribute the density in the form which minimizes the total energy and corresponds then to a variational procedure in the formulation of the theory. In the definition above if $\mathbf{kin}(\mathbf{x})+\mathbf{V}(\mathbf{x})+\mathbf{V}_{xc}(\mathbf{x})+\varepsilon_0(\mathbf{x})$ are properly defined, then we should require that $\int\rho(\mathbf{x})\varepsilon_0(\mathbf{x})d\mathbf{v}=0$. We now recover the Hohenberg-Kohn Theorems [48] and the Kohn-Sham minimization procedure [84] for the definition of this two terms

$$\delta\big(E\left[\rho\right]-\varepsilon\left\{\int\rho(\mathbf{x})\,d\mathbf{v}-1\right\}\big)=0 \quad , \tag{23}$$

allows the direct self-consistent determination of $\rho(\mathbf{x})$ and ε [30].

In the formalism we have a very useful self-consistency relation

$$\begin{aligned}\Psi(x)=\int w(\mathbf{x})G(\mathbf{x},\mathbf{x}')\Psi(\mathbf{x}')+\Psi_0(\mathbf{x}) \quad \text{where}\\ w(\mathbf{x}) \;=\; \Delta\rho(\mathbf{x})\varepsilon+\rho(\mathbf{x})\boldsymbol{\Delta}\varepsilon(\mathbf{x}) \quad .\end{aligned} \tag{24}$$

We have introduced both, the response function of the system $G(\mathbf{x},\mathbf{x}')$, and the effective potential which would be caused either by fluctuations of the density or by differences in the local definition of energy per particle. This reaction would propagate to all points of the distribution to achieve stability. We have used elsewhere an approximation for the response function $G(\mathbf{x},\mathbf{x}')$ in terms of the lowest elementary excitations of a gas of non interacting particles.

3.4. Recovering Standard DFT

The auxiliary function Ψ in (18) is identical to the standard quantum mechanical ψ in the case of a “one” particle system, otherwise we should consider it as the function which represents the gauge dependent square root of the density of action, obeying the equations

$$h_{eff}\Psi(\mathbf{r}) = \mu\Psi(\mathbf{r}) \quad , \tag{25}$$

where we have defined an effective operator

$$h_{eff} = -\nabla^2 + \upsilon(\mathbf{r}) + \upsilon_{eff}(\mathbf{r}) \quad , \tag{26}$$

such that the system's energy

$$T_s[\rho] = \int \Psi^\dagger(\mathbf{r})(-\nabla^2)\Psi(\mathbf{r}) + T_\theta[\rho] \quad , \tag{27}$$

defining, variationally, the effective potential

$$\frac{\delta T_s[\rho]}{\delta\rho(\mathbf{r})} = -\frac{\Psi^\dagger\nabla^2\Psi}{\Psi^\dagger\Psi} + \upsilon_\theta([\rho];\mathbf{r}) + V_{ss}([\rho];\mathbf{r}) \quad , \tag{28}$$

$$\upsilon_\theta([\rho];\mathbf{r}) + V_{ss}([\rho];\mathbf{r}) = \frac{\delta T_\theta[\rho]}{\delta\rho(\mathbf{r})} \quad , \quad \frac{\delta T_s[\rho]}{\delta\rho(\mathbf{r})} + \upsilon_{KS}([\rho];\mathbf{r}) = \varepsilon_M \quad . \tag{29}$$

then (25) reads

$$-\nabla^2\Psi + \upsilon_{KS}([\rho];\mathbf{r})\Psi + \{\upsilon_\theta([\rho];\mathbf{r}) + V_{ss}([\rho];\mathbf{r})\}\,\Psi = \mu\Psi(\mathbf{r}) \quad . \tag{30}$$

Now we quote something which should in fact be a result of some considerations of the next section. From the definition of auxiliary functions below, we obtain for the last term:

$$\Delta\bar{\varepsilon}(\mathbf{r})\rho(\mathbf{r}) = \sum_i \varepsilon_i \left|\psi_i(\mathbf{r})\right|^2 - \bar{\varepsilon}\rho(\mathbf{r}) \quad , \tag{31}$$

which is equivalent to the optimization of

$$\Omega = E[\rho] - \mu\left[\int \rho(\mathbf{r})d\mathbf{r} - \mathbf{N}\right] - \lambda\left[\int \sum \varepsilon_i \left|\psi_i(\mathbf{r})\right|^2 d\mathbf{r} - \int \bar{\varepsilon}\rho(\mathbf{r})d\mathbf{r}\right] , \tag{32}$$

using a set of auxiliary functions ϕ_i, with the index i running through all possible forms of extracting one particle from the system

$$-\nabla^2\phi_i + [\upsilon_{xc}(\mathbf{r}) + \upsilon_{Coul}(\mathbf{r}) + \upsilon_{ext}(\mathbf{r})]\phi_i = \varepsilon_i\phi_i \quad . \tag{33}$$

which will define the elementary excitations of the system corresponding to the removal of one particle with rate of change of the energy ε_i

$$\sum_i h_i(\mathbf{r})^{KS} = \sum_i \varepsilon_i \left|\phi_i\right|^2 \quad , \tag{34}$$

this is equivalent to the Kohn–Sham procedure and the use of the Kohn-Sham effective Hamiltonian $h_i(\mathbf{r})^{KS}$

$$h_i(\mathbf{r})^{KS} = -\phi_i^* \nabla^2 \phi_i + \phi_i^* [\upsilon_{xc}(\mathbf{r}) + \upsilon_{Coul}(\mathbf{r}) + \upsilon_{ext}(\mathbf{r})]\phi_i \quad . \tag{35}$$

and then for the gauge dependent square root of the density auxiliary function [30]. Corresponding to (25) we have the final equation

$$-\nabla^2 \Psi + \upsilon^{FK}(\mathbf{r})\Psi = \bar{\varepsilon}\Psi \quad , \tag{36}$$

where we have defined the effective potential

$$\upsilon^{FK}(\mathbf{r}) = \upsilon_{xc}(\mathbf{r}) + \upsilon_{Coul}(\mathbf{r}) + \upsilon_{ext}(\mathbf{r}) + \Psi^\dagger \nabla^2 \Psi + k(\mathbf{r}) - \Delta\bar{\varepsilon}(\mathbf{r}) \quad . \tag{37}$$

The last three terms correspond, the first two to the correct kinetic energy density and the last one, as above, the symmetry constraint potential arising from the actual values of the energy necessary to remove one electron from the system and the average energy per particle:

$$\Delta\bar{\varepsilon}(\mathbf{r}) = \sum n_i \varepsilon_i \left|\phi_i(\mathbf{r})\right|^2 / \rho(\mathbf{r}) - \bar{\varepsilon} \quad . \tag{38}$$

3.5. From Action Density Functional to Wave Mechanics

We now proceed formally to show that the procedure within START described here is equivalent to the usual postulation of the principles of the wave equation approach to quantum theory.

(1.) We have defined, for the representation of the physical system an (complex function) adimensional density of action $\mathbf{K}(\mathbf{X})$ at each space-time point $\mathbf{X} = \boldsymbol{\gamma}_\mu X^\mu$.

(2.) The action is factorized, for its study, into a density $\mathbf{n}(\mathbf{X})$ and a local average action per particle $\mathbf{k}(\mathbf{X})$

$$\mathbf{K}(\mathbf{X}) = \mathbf{n}(\mathbf{X})\mathbf{k}(\mathbf{X}) \quad . \tag{39}$$

(3.) The energy of the system $E(\mathbf{X}) = i\hbar\partial\mathbf{K}(\mathbf{X})/\partial t$ is obtained from this action density or in general

$$\Pi_\mu(\mathbf{X}) = [i\hbar\partial\mathbf{K}(\mathbf{X})/\partial X^\mu]_R = \mathbf{n}(\mathbf{X})\mathbf{p}_\mu(\mathbf{X}) \quad , \tag{40}$$

for the energy momentum four vector $\mathbf{\Pi}$. This equation is in fact the defining equation for $\mathbf{K}$.

(4.) To see the correspondence to the space-time-action geometry (STA) we remind the reader that the fifth axis of this geometry was labelled by $i\gamma_5$ and that γ_5 is the unit four-volume in spacetime, then the density

$$\mathbf{K}(\mathbf{X}) = (\mathbf{i}/\hbar)\mathbf{A}(\mathbf{X}) = -\mathbf{a}(\mathbf{X})\mathbf{i}\boldsymbol{\gamma}_5\gamma^5 = -\mathbf{i}\mathbf{a}(\mathbf{X}) \quad , \tag{41}$$

per unit spacetime volume. We see that this quantity is a pure imaginary complex number. Note that action acquires a negative sign from $(i\gamma_5)(-i\gamma_5) = (\gamma_5)^2 = -\mathbf{1}$

(5.) This action distribution representing the physical system has two sources of gauge dependence. (a) the dependence on the definition of the reference spacetime hypersurfaces in the STA space, a dependence related to gravitation, and (b) the dependence on the arbitrary (either by incomplete knowledge or by practical decision) choice of the type(s) of field(s) whose density is represented by $\mathbf{n}(\mathbf{X})$ and, by definition of the fields, their energy contributions. Here we should consider from very simple, one type of action carriers, to complicated cases like a system of an electron e^- and a W^+ (which could also be a neutrino ν), where the description of the system should include the complete range of possibilities. To solve this description problem we now introduce a description and gauge dependent auxiliary function

$$\Psi(\mathbf{X}; \{\mathbf{x}_i, t_i;\, i = 1, ..., n\}) \quad , \tag{42a}$$

which allows to write for the action density

$$\mathbf{K}(\mathbf{X}) = \overset{\{i\}}{tr}\ \hat{A}(\Psi(\mathbf{X})\Psi^\dagger(\mathbf{X})) \quad , \tag{43}$$

which for a single element $n = 1$ is

$$\mathbf{K}(\mathbf{X}) = \hat{A}(\Psi(\mathbf{X})\Psi^\dagger(\mathbf{X})) \quad . \tag{44}$$

(6.) The self-consistent properties of $\Psi(\mathbf{X}; \{\mathbf{x}_i, t_i;\, i = 1, ..., n\})$ from (19)-(24) are then

a.) From the simplest case, that of the homogeneous distribution of action with an assumed single carrier, where (defining $\mathbf{k} = \mathbf{\Pi}/\hbar$)

$$\Psi(\mathbf{X}) = \boldsymbol{\rho}^{1/2} e^{-i\mathbf{k}\cdot\mathbf{X}} \quad , \tag{45}$$

and therefore

$$\Pi_\mu = [\Psi^\dagger i\hbar\partial_\mu\Psi]_R = \hbar\rho k_\mu = \rho\mathbf{p}_\mu \quad . \tag{46}$$

Now, defining the auxiliary reference energy-momentum m

$$m = (\Pi^{\mu}\Pi_{\mu})^{1/2} \quad , \tag{47}$$

then

$$D_0 D_0 \Psi = m^2 \Psi \quad , \tag{48}$$

where we have used the notation, in space-time-action geometry

$$i\hbar\gamma^{\mu}\partial\mu\Psi = D_0\Psi = m\Psi \quad , \tag{49}$$

which shows that the auxiliary function Ψ has the same initial properties as the standard wave function in quantum mechanics. It is now immediate that the description freedom corresponds to the gauge theory approach, where the "Lagrangian" density is proportional to $\mathbf{K}(\mathbf{X})$, the Ψ are gauged and the D_0 operator is enlarged to the covariant derivative D, to keep $\mathbf{K}(\mathbf{X})$ gauge invariant.

b.) In the case of several identical among themselves carriers we can now construct the Ψ as $\Psi = \overset{i}{\Pi}\, \phi_i(\mathbf{x}_i, t_i)$ and define the potentials $\mathbf{V}$ and $\mathbf{V}_{xc}$ accordingly or use a more complicated expression for Ψ and a (simpler) expression for $\mathbf{V}$ and $\mathbf{V}_{xc}$. Otherwise a more complicated, interacting particles, definition of Ψ and $\mathbf{V}$ should contain the sum of the interparticle interactions. See next section.

c.) The case of several types of carriers corresponds to (sums and) products of descriptions of type b).

d.) The use of a description of an evolving system with changing types of carriers defines interaction Lagrangians where the sum of products of descriptions of type b) are used to represent our uncertainty in the actual distribution of action, keeping nevertheless the $\mathbf{K}(\mathbf{X})$ invariant.

All these descriptions obeying the principles above, show the intrinsic connection between the postulates of START and the structure and interpretation of wave quantum mechanics.

3.6. Description in Terms of Interacting Particles

The basic description in terms of particles (particle fields in practice) is that of N interacting particles:

1) Each point of the particle field i is endowed of a self energy, expressed as its mass m_i, and spin s_i and a collection of charges $\{q_i^{(g)}\}$, one for every gauge field g,

2) all particles i are subjected to local external potentials $V_i(\mathbf{x})$ representing the "rest of the universe" effects where

$$V_i(\mathbf{x}) = m_i V^{grav}(\mathbf{x}) + \sum_g q_i^{(g)} V^{(g)}(\mathbf{x}) \quad , \tag{50}$$

3) all the N particles i are supposed to have an independent particles kinetic energy contribution $\mathbf{kin}_i(\mathbf{x})$
4) there is an interparticle pairwise potential energy $(i \neq j)$

$$\sum_g \frac{1}{2} \sum_i \sum_j q_i^{(g)} q_j^{(g)} f\left(\mathbf{u}_{i,} \mathbf{u}_{j;} r_{ij}\right) \quad , \tag{51}$$

which is proportional to the products of the charges and to a function of the distances r_{ij} between points of the fields related to the particles, the basic example being the electromagnetic case $q_i^{(e)} q_j^{(e)} (\mathbf{u}_i \cdot \mathbf{u}_j/c^2)/r_{ij}$,where for particles at relative rest $(\mathbf{u}_i \cdot \mathbf{u}_j/c^2) = 1$, this being the fundamental **definition** of an interaction as a force which decays as the surface of a sphere with the source at the center, that is the capability of a source to do a work is constant on the surface of spheres and proportional to the source strength.
5) Other interaction terms, depending on the masses or spins of the particles. Until now there has been no practical use of more complicated terms, like terms depending on products of charges of different gauge fields $q_i^{(g)} q_j^{(g')}$. In practice, for non elementary particles, three body terms and "effective" charges have been used, this being perhaps a guide to establish that a particle is not elementary. When r_{ij} is large, with respect to a measure of the extent of the distribution of both the i and the j field, a center of distribution (equidistant) r_{ij}^o can be used in practice, fact that allows to consider the fields i and j as point-like objects.

The use of (at least) 1), 2), 3) and 4) induces either
a) the use of a non local $\Psi_{non-local} = \Psi_{non-\ell ocal}(\{\mathbf{x}_i; i = 1, ..., N\})$ or
b) the local formulation obtained by the introduction of this non locality as a self consistent local potential (which requires for its calculation the knowledge of $\Psi_{n\ell}$, considered now as an auxiliary calculation procedure).

Case a) corresponds to standard quantum mechanics where the auxiliary function Ψ is constructed as sums of products of sums of basic functions. The last sum corresponds to an assumed distribution in space. The products correspond to considering a set of those sums as an independent particle field scheme and the first sum to account both for the statistics of the auxiliary fields and for all possible forms of response of the system to the possibility of removing one particle from it.

Case b) corresponds to keeping the local $\Psi(x)$ and introducing the result of the non local interaction as corrections to the kinetic energies obtained from $\psi(x)$ and as an equivalent, average, local, interparticle potential where also the effect of the statistics and of the full response of the system are included. This last term is well known [84] as the local exchange-correlation potential in standard DFT. The remaining term was introduced [30] in the study of $\Psi(x)$ for a many electron system.

This is not the only set of possibilities, a third major line of approach has been developed mainly in connection with high energy physics and the study of elementary particles. It consists in formulating an independent particle approach using an action related to the local effect of the gauge fields into the particle fields through terms

$$q_i^{(g)}\mathbf{A}_{(g)}(\mathbf{x}).d\mathbf{x}_i \quad , \tag{52}$$

the scalar product of the vector $\mathbf{A}_{(g)}$ and the vector $d\mathbf{x}_i$, or $\mathbf{u}_i =.d\mathbf{x}_i/dt$ if energy is computed. This is achieved at the expense of allowing independent existence to the gauge interaction fields. This has the advantage of allowing the possibility of describing the gauge fields independently of the source or target particle fields, introduce the quantization of this gauge interaction fields which carry energy-momentum, spin and geometrical information of the possible source or target fields.
The gauge interaction fields are assigned a gauge independent field strength

$$F_{\mu\nu}^{(g)} = \partial_\mu A_\mu^{(g)} - \partial_\nu A_\mu^{(g)} \quad , \tag{53}$$

and an **action** of the gauge field itself

$$a_{(g)}(x) = -\frac{1}{4}F_{(g)}^{\mu\nu}(x)F_{\mu\nu}^{(g)}(x) \quad , \tag{54}$$

to be added to the particle's field action. The sum is a local action and a local energy-momentum by consequence. The pass from case 1) to case 3) is straightforward by partial integration using a source equation

$$A^{(g)}(x) = \sum_{i\neq test} \nabla^2 j_i^{(g)}(x) \quad . \tag{55}$$

But it must be stressed that the energy related to (54) requires in general the integration over volumes much larger than those of the integration of $\rho_i(x)$. Then in case 1) the energy related to external sources of gauge fields should be added, because only the action related to the system of particles $\{i\}$ has been included.

There are many technical difficulties in this approach, which is in principle quantum field theory based in quantum electrodynamics as the simplest case and in the Maxwell theory in the classical formulation, some of them would disappear if a hybrid approach is taken, using 1) and allowing for (54) for the description of the external influences. The problems related to the non-abelian character of the gauge fields would require nevertheless the use of the special mathematical techniques now in use in the standard model of elementary particles. This considerations do not apply to the use of DFT to the many electron system, the most common example. Pairwise interactions and gauge fields are equivalent dual formulations which should be explicitly followed.

The description here developed requires, at least, the analysis of two particular cases:

1) the consideration of the relativistic limit

2) the consideration of large interparticle separations.

In the first case the current j_μ is given by ρu_μ with $u^2 = u^\mu u_\mu = c^2$, then for the description of the dynamics the energy density is given not by the sum of the kinetic energy and the mass energy $(m_0c^2)_i$ but by the energy-momentum m_ic^2 and the interaction energy

$$\sum_{i<j} \frac{q_iq_j}{|\mathbf{x}_j - \mathbf{x}_i|} \frac{\mathbf{u}_i{\cdot}\mathbf{u}_j}{c^2} \quad . \tag{56}$$

That is, from (56), each particle is a test particle in respect to the others. Then for each (test) particle j we can define and "external" electromagnetic field

$$A_\mu^{(j)}(\mathbf{x}) = \sum_{k\neq j} \int \frac{\rho_k(\mathbf{x}_k)q_k}{|\mathbf{x} - \mathbf{x}_k|} \frac{u_\mu(\mathbf{x}_k)}{c} \quad , \tag{57}$$

completely defined by the set of distributions $\{\rho_k(\mathbf{x}_k),\, q_k,\, u_\mu(\mathbf{x}_k);\text{ all } k\}$, that is: there is no need to consider new degrees of freedom for the $A_\mu^{(j)}(\mathbf{x})$, and these quantities depend on the sources not on the test particle (one at a time). In fact some test particles could have zero charge.

The second point is related to the interaction between two, in relative motion, carriers or more when the final action can be described by a similar system of carriers but with n units of angular momentum have been changed $\Delta l = \pm nh$ and a corresponding energy change $\Delta e = \sum_m n_m h\nu_m$ with $\sum_m n_m = n$. This **real** processes correspond then to quantized emission or absorption of energy; there is nothing in the basic principles forbidding these processes and they obey the third principle. The puzzling fact is that the processes can be described by $A_\mu(\mathbf{x})$ fields also and then the particles in the interaction picture above is transform into a dual description where the gauge fields can also acquire physical existence. The set $\{n_m\}$ is not a conserved set of quantities in the presence of carriers, as far as all possibilities to describe action have to be included. We are then forced to systematically enlarge the formalism to include these possibilities. This accounts for the fact that experimentally we can produce light and handle its energy by interactions with the appropriate set of carriers. Otherwise, for lecturing purposes, we are familiarized with a picture where this energy quanta $h\nu_m$ carrying angular momentum h, is considered a sort of independent light beam which is deflected or reflected or refracted without loosing the identity of the initial quanta, this is not what experiment supports because the description of reflections and refractions is, at the fundamental level the description of absorption and re-emission of the quanta.

3.6.1. A Note on the Probabilistic Interpretation of Ψ.

Because the auxiliary functions describing the action contributions will either appear as products of functions $\varphi_i\varphi_2\ldots$ or as sums of functions $\phi_1+\phi_2+\ldots$ the use of derivatives D as operators originate both a probabilistic interpretation and, in fact as a consequence, a systematic method to obtain Ψ.

In fact for a product $\varphi_1\varphi_2$, because $D\,(a\varphi_1\varphi_2) = a[(D\varphi_1)\varphi_2 + \varphi_1(D\varphi_2)]$ the energy-momentum contributions will appear as sums of independent terms. Also, because $D(a\phi_1 + b\phi_2) = aD\phi_1 + bD\phi_2$, the energy contributions from a sum of functions appears as a weighted sum of dependent contributions. This is typical of probability theory and a probabilistic language will faithfully be useful to describe the total action.

An additional probabilistic concept, different from the one described above, arises from the algebra of the operators themselves, because the action being $x^\mu p_\mu$, and its operator $\hat{a} = x^\mu\hat{p}_\mu = x^\mu(i\hbar\partial_{x^\mu})$, then we, from the chain rule for derivatives, obtain the operator

$$[x^\mu, \hat{p}_\mu] \equiv x^\mu\hat{p}_\mu - \hat{p}_\mu x^\mu = i\hbar \quad , \tag{58}$$

with the well known Heisenberg limitation, introducing an uncertainty in our possibilities to know (not the action but) the factors of the action, separately, for a given action distribution, up to the small but highly significant value of $\hbar$. Because this is a fundamental restriction on the description of the action distribution as that of point like carriers this uncertainty is presented as a basic property of matter, independent from our choice to describe the matter.

We can then conclude that the action distribution in spacetime description of matter agrees, without actual limitations, with our present experimental and theoretical knowledge of matter and interaction fields.

In space-time-action geometry the main dynamical principle is that all trajectories should be minimal, then defining the (square of the) differential $dS^2 = ds^2 - (da)^2$, where $dS^2 = g_{\mu\nu} dx^\mu dx^\nu$ is the spacetime differential and $(da)^2$ the action differential. In a first, non united geometry, approximation the minimal principle

$$\delta(dS^2) = 0 \quad , \tag{59}$$

can be separated into the kinematical principle of (general) relativity

$$\delta(ds^2) = 0 \quad , \tag{60}$$

and the principle of minimum action

$$\delta(da^2) = 0 \quad . \tag{61}$$

For some phenomena, light as the main example, (60) and (61) are separately obeyed given that $(cdt)^2 - (dx)^2 = 0$ and $g_{\mu\nu} p^\mu x^\nu = 0$ because $g_{\mu\nu} = diag(1, -1, -1, -1)$ and $\varepsilon = pc = h\nu = hc/\lambda$. Otherwise the principle of minimal action is universally accepted in the formulation of physical principles. If (59) is accepted a geometrical model for mass appears in our theory [61].

In relation to our construction of the auxiliary function Ψ we can then use a trial set of contributions to the action

$$\Psi_t = \sum_i a_i \left(\Pi_{j \in \{j\}_i} \varphi_j \right) \quad , \tag{62}$$

with the φ_j also a composition of functions representing some contributions to the action. A chain derivative

$$\sum_i \frac{\partial \Psi^\dagger}{\partial a_i} \frac{\partial E}{\partial \Psi^\dagger} = 0 \quad , \tag{63}$$

will allow the optimization of the description, then a variational principle for energy exists which is, in Action Density Functional Theory, the equivalent to the Hohenberg-Kohn Theorems [48].

3.7. Some Remarks about the Formalism

We have presented an action-density functional formalism, developed it and shown that not only the standard density functional theory is recovered and that in a sense it is more fundamental than wave function wave mechanics, but also that the analysis of the mapping of the density matrix into a density allowable for density functional theory [59], requires the introduction of auxiliary terms which represent the internal symmetries of the system.

Several basic principles of quantum mechanics are then natural structures in the approach developed here to describe matter as a distribution of action in space-time (energy distribution over space).

In the separation of carriers discussed here the interaction of a carrier with itself is to be ruled out, in the form called self-exchange. If a diagrammatic procedure is used, diagrams relating a particle with itself are to be ruled out. Nevertheless in a system a "real" interaction particle can be created as a result of energy transfer in a pairwise process and the emitted particle could be reabsorbed by any one of the other carriers present in the system, this real interaction has to be included.

3.8. Curvature

The geometrical union of phase space to a carrier space, above, allows a rederivation of the proportionality between stress-energy-momentum and curvature.

There are at least two approaches to the Riemann curvature tensor. One is space intrinsic and the second considers the notion of an extrinsic normal to the space. The corresponding formulas are, for the intrinsic formulation

$$R^{\delta}_{\alpha\beta\gamma} := \partial_\beta \Gamma^{\delta}_{\gamma\alpha} - \partial_\gamma \Gamma^{\delta}_{\beta\alpha} + \Gamma^{\delta}_{\beta\nu}\Gamma^{\nu}_{\gamma\alpha} - \Gamma^{\delta}_{\gamma\mu}\Gamma^{\mu}_{\beta\alpha} \quad , \tag{64}$$

the definition of the Riemann (-Christoffel) curvature tensor in terms of the Christoffel symbols, and, for the second approach,the Gauss identity

$$R^{\delta}_{\alpha\beta\gamma} = P^{\delta}_{\beta} P_{\alpha\gamma} - P^{\delta}_{\gamma} P_{\alpha\beta} \quad , \tag{65}$$

derived from the Leibnitz rule

$$\frac{\partial \mathbf{x}}{\partial x_\gamma \partial x_\beta \partial x_\alpha} = \frac{\partial \mathbf{x}}{\partial x_\beta \partial x_\gamma \partial x_\alpha} \quad , \tag{66}$$

using the embedded curved spaces quadratic form, where a normal $\mathbf{n}$ is considered,

$$P_{\alpha\beta} = (\frac{\partial \mathbf{x}}{\partial x_\beta \partial x_\alpha}, \mathbf{n}) \quad , \tag{67}$$

scalar product which computes the $\mathbf{n}$ component of the variation of a vector $e_\alpha(x)$ when translated by dx_β in the $e_\beta(x)$ **direction.** This definition of $R^\delta_{\alpha\beta\gamma}$ is then based on the assumed existence of a normal $\mathbf{n}$ to a space $M \in \mathbf{x}$. **This is otherwise, precisely, the consequence of the introduction of the action coordinate in a form equivalent to the complexification of a space, a normal exists at every point.**

As the normal direction to spacetime has the dimension of action, then (67) corresponds to computing the rate of change of action, that is energy-momentum, in relation to the local curvature of spacetime. This is the physical content in START of the fundamental Einstein formula for general relativity,

3.9. Matter Presented as Geometry and the Minimal Trajectory Condition

The procedure goes through the following steps:

1.- Assume space and time geometrically unified in Minkowski spacetime manifold **M**

2.- Provide it with a full geometric structure

3.- Introduce action as a 5^{th} carrier space coordinate. Obtain its full geometry and, from the initial considerations of this paper, show that it is equivalent to the complexification of spacetime (4-D transforms into 5-D geometry).

4.- Project out a physical (in general curved) local **M** manifold

5.- The projection has Induced Energy Density from the considerations of the previous section above. In fact an **action density** has been defined at every point of **spacetime**, but for a given observer time passes continuously and it observes the rate of change of action, that is an energy density in space.

6.- Provide a stable structure to the energy density through a chiral constrain to the massless matter fields, this projection is described below in the section referring to matter and interaction fields. This implies that when a pair of particles are created from a given amount of energy the symmetries of the particles should add to zero: $\Delta q = q_1 + q_2 = 0$, $\Delta \mathbf{s} = 0$ for example, other symmetries are known in the literature as interval symmetries. Also when energy is transferred from one carrier to the other the intermediate carrier should contain the change in symmetries of the first and produce changes in symmetries of the second, the best known example is the spin of the photon, $\pm\hbar$, taken or given to the carriers.

7.- The procedure generates then charges and, through gauging, interaction fields

8.- Every point of the matter fields is a source of a geometric wave, from the continuity of the distribution of action in space time, required by assuming analyticity.

9.- As we have discussed elsewhere the resulting structures posses all the symmetries of the (enlarged) Standard Model

This systematic procedure is followed in detail in Keller 1999, where it is shown that the geometric structure of the Space-Time-Action geometry allows, through a Kaluza-Klein like mechanism, the generation of energy density and of energy carrying interaction fields. When this geometric structures self-stabilize, with chiral symmetry constrains, a collection of stable matter and interaction fields is obtained with the symmetry of families of elementary fields with the Standard Model content.

Before entering to the formal presentation of the theory of the electron we must mention that we have presented elsewhere the construction of a theory of lepton and quark fields (Keller 1981-1999) using chiral geometry for the formulation of the multivector generalization of the Dirac factorization of the four dimensional d'Alembert operator $\nabla^2 = \partial^\mu \partial_\mu$, written in the Lorentz invariant form

$$D_{(d,f)} = \Gamma^{\mu}_{(f)} \partial^{(d)}_{\mu} \quad , \tag{68}$$

in order to show the relation to the Dirac's original factorization in the simplest possible form.

In (68) the $\Gamma^{\mu}_{(f)}$ are either the fundamental or generalized (reducible representation) Dirac γ^μ matrices. We have shown [64] that these fields constitute a set with all the known properties of an elementary particle's family, the fields they represent are: massless or massive after interactions are considered and charged (integer or fractional).

There the collection of the constructed fields have weak charge and color, and in general the characteristics usually postulated on phenomenological basis like composites being colorless, confinement, etc. here being immediate consequences of the defining equations.

The principal change from the standard model is that we are dealing there with a **theory** where the equations have, as constitutive parts, a series of conditions reproducing what the phenomenological approach showed to be necessary. The conditions are related to the basic properties of spacetime as a frame of reference to describe physics.

Because of the appearance or not of the $i\gamma^5$ factors, the fields have definite chiral properties. Only the electron field in the theory may have both chiralities simultaneously and therefore can be, as a free field, massive, charged (reference charge ± 1) and weak charged: this is identified as the electron field. The resulting theory is a **chiral geometry** theory of charge, isospin and color. The theory has a **Lagrangian formulation** that reproduces all aspects of the standard theory [52, 78]. The Higgs mechanism has a purely geometric character in the present analysis [85]. The masses of the densities of the elementary particle fields are all expressed in terms of the electron mass m_0.

Confinement results, within the theory, from the requirement that the Lorentz symmetry should not be broken even at local level. The same requirement gives rise to the colorless condition for hadrons, the new feature is that hadrons should be both globally and locally colorless. Fractional charges are also a natural consequence of the gauging properties of the Lagrangian.

Allowing the metric to be $g_{uv} = \mathbf{diag}(1, -1, -1, -1, 1)$ then the units used for the densities are such that for the electron and for any massless fields including the photon $dS^2 = 0$, then this basic fields are the simplest geometrical trajectories in START.

3.10. The Structural Consequences of Using Complex Spacetime Algebra

Here we first refer to the appendix for the geometrical analysis.

Besides generating the (local) Standard Model, the use of complex spacetime algebra has relevant consequences for large scale or small scale physics.

The large number of possibilities open in START are derived from the fact that we have now both: .- a five dimensional base geometry .- and for the analysis of the matter and interaction fields a natural form to introduce chiral symmetry as a guiding rule. We will present the example of the neutrino field below.

The STA geometry (void of matter) is assumed to have a Ricci tensor: $a \cdot \mathbf{R}(a \wedge b) = R(b) = 0$ where $\mathbf{R}$ $(a \wedge b)$ is the curvature $\mathbf{R}(a \wedge b) = a \cdot \nabla\Omega(b) - b \cdot \nabla\Omega(a) + \Omega(a) \times \Omega(b)$ with the bivector functions $\Omega((b), x)$ being the local Lorentz group connections $D_a B = a\nabla \cdot B + \Omega(a) \times B$ in the (curved or flat) space of $G(R^5; \mathcal{C}\updownarrow_{\in,\ni})$. The corresponding 5- dimensional (flat space) Einstein tensor would be $G_{AB} = e_A \cdot (R(e_B) - Rg_{AB}e_A) = 0$. From Campbell (1926) any analytic $N-1$ dimensional manifold can be nested in a N dimensional flat manifold $R(b) = 0$. We use the well known procedure of dividing the$5-D$ metric tensor G_{AB} into a geometric part and an induced energy part. We obtain, by direct substitution the components of $G(a)$ rewritten as geometry and energy: in the form $G(a) = \kappa T(a)$ with $T(a)$ in the canonical form $e_\mu \cdot T(e_\nu) = (p + \rho)u_\mu u_\nu - pg_{\mu\nu} \quad \mu,\nu = 0, 1, 2, 3$. The fifth dimension has been introduced as a geometrical consequence of the complexification. The complex spacetime line element becomes $dS^2 = g_{\mu\nu}dx^\mu dx^\nu + G_{55}(\frac{G_{5\mu}}{G_{55}}dx^\mu)^2$ with the spacetime standard line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ acquires the necessary values to represent different physical conditions of the system. Identification of the energy density and electroweak potential $A_\mu = \frac{G_{5\mu}}{G_{55}}$ to obtain $dS^2 = g_{\mu\nu}dx^\mu dx^\nu - g^2(dx^4 + A_\mu dx^\mu)^2$ rewritten as $dS^2 = g_{\mu\nu}dx^\mu dx^\nu - \frac{f^2}{g^2}dS^2$; $(1+\frac{f^2}{g^2})dS^2 = ds^2$. The definition of g^2 above corresponds, in geometric algebra to the scalar product of a multivector with itself. In the auxiliary five dimensional space we could set

$$G_{\mu\nu} = g_{\mu\nu} - g^2(k_\mu k_\nu - \frac{e^2}{g^2}A_\mu A_\nu) \quad . \tag{69}$$

Here the square of the action appears as the sum of a mass terms and an interaction fields term (see theorem in appendix).

There is a straightforward procedure to show the physical implications of using STA geometry:

1. We consider a simple solution of the Einstein equations in the presence of dust matter.

2. We introduce complex coordinates in our geometric space, as described above, and show that the Einstein equations are a condition for the relationship between the real part and the imaginary part of the line element.

3. We transform the equation obtained in the second step to a condition for 5-D flat space corresponding to the complex geometry system. We obtain, from this step, a constant which fixes the metric relation between the real and the imaginary part of the coordinates. Because the conditions derived in the previous item this constant appears as a basic property of matter, which in our approach is related to κ_0 in (10)

3.11. Some Remarks

The use of the space-time-action (STA) geometry, isomorphic to complex spacetime, results in a five dimensional geometry, and allows the construction of a generalization of the Kaluza-Klein theory (see, for example [120]) with both induced matter and interaction fields and the new features consisting in a natural existence of the $SU(3) \otimes SU(2) \otimes U(1)$ theory for the elementary particles fields.

We have obtained that in START:

.- The formulation allows the projection of a (flat or curved) spacetime. In the case of curved space time the term which compensates for the curvature is an induced energy term.

.- The separation of zones with induced energy requires the stabilization through the creation of symmetry constrains for each region (elementary particle field) which can not be annihilated without annihilation of a second or more regions with compensating geometric constrains.

.- The natural structures in this geometry are associated with a set of equations for massless particles allowing a series of factorizations of the Laplacian operator and associated Dirac-like equations, this set of related equations generates 3 families of elementary particles with the experimentally observed lepton and quark content for each family and the experimentally observed electroweak color interactions and other related properties

.- This requires the use of gauge transformation of the different fields and their relative gauging properties. The factorizations used in [53, 54, 55, 64] $\nabla^2 = (\Gamma^{\mu}_{(f)}\partial^{(d)}_{\mu})\bullet(\Gamma^{\nu}_{(f)}\partial^{(d)}_{\nu})$ and the related Dirac-like equations $\Gamma^{\mu}_{(f)}\partial^{(d)}_{\mu}\psi_{(d,f)} = 0$ were studied and their symmetries were given. The analysis showed that $\Gamma^{\mu}_{(f)}$ generate the 3 families, the $\partial^{(d)}_{\mu}$ generate the observed lepton and quark content of the families, this is a result of the inherent symmetry restrictions introduced by (d, f) in the equations and in the wave functions, from the gauging of both the equations and the wave functions. For the electron $\nabla^2 = D_0 D_0$. For the neutrino see below.

The standard model (SM) is derived from an analysis of wave equations in spacetime, using chirality as a basic symmetry.

In contrast to the usual approach to SM, the properties for the different fields of the model are consequences of the relative properties of the equations, among themselves and in relation to spacetime, and therefore, they do not need to be postulates of the theory. Also we show that the formulation includes all possibilities open with higher dimensional geometries, including the gauging of the geometry to generate (a gauge theory) gravitation and in fact what has been called theories of induced matter and charge.

In the theory we present here the physical properties are now a constitutive part of the wave equations. The relative properties are clearly shown when supermatrices describe a collection of fields. Off diagonal terms couple them among themselves.

STA and its T_M (complex) allow enough degrees of freedom to construct a theory of elementary particles and their interactions. Specially important is that all known interactions are properly described. No additional isospin space is therefore needed, it is generated by the relative properties of the fields, the same applies to the color space. Nucleons like proton or neutron and mesons are, within this theory, composite fields but elementary particles. In fact these composite "elementary" particles cannot, even if enough energy is available, be split into smaller components; the requirement of rotational invariance forces the "colorless" combination of quarks, even to the smallest possible experimental probe size

Then START allows a compact formulation of several basic principles of physics.

4. The Theory of the Electron

In **ET1,** the first part of this paper, we already stated that there is no doubt that the Dirac equation for the electron provides a sound starting point for the theory of the electron field both in the presence of electromagnetic and of weak interactions. We have also shown elsewhere that a generalization of the relativistic, spinor, equation for the field may account successfully for most of the known properties of the matter and their interaction fields. The interaction fields being obtained as gauge fields of the matter field. Here we proceed to construct a theory of the electron from START and, because the Dirac equation is obtained from the first steps, the development of the theory will mostly appear as a theoretical derivation of the standard theory where the geometrical characterization of the different concepts (postulated in the Dirac theory) defines them and, most important, limits their usage.

For completeness we present in appendix B, a review of our knowledge of the electron theory and the interpretation of the electron properties. There, after some general considerations, we first briefly remind of the work of Fock and Ivanenko in relation with the geometric content of the electron theory, and of the discovery of the use of Clifford algebras to study the Maxwell field and comment on the well known geometric algebra formulation of the Dirac theory presented by Hestenes since 1967. Including some remarks on the analysis of the French school, Casanova, Boudet, Quilichini. We show that this corresponds to a Cartan mapping, inverse Cartan mapping and moreover we show that the Maxwell equations can be faithfully put into a Dirac form and that an inverse Cartan mapping can also be applied to the Maxwell field equations. We review the analysis independently started by Daviau in 1989 and Campolattaro in 1990 where they show a full mathematical equivalence of the Maxwell and Dirac equations. It is remarked that there is no "equivalence of Dirac and Maxwell fields..." as can be thought from the mathematical identity of the fundamental equations. That this mathematical equivalence could exist has been thoroughly discussed by Rodrigues, Vaz and Recami (1993), Rodrigues and Vaz (1994) but in fact although the equations can be written in exactly the same basic form, the solutions have further constrains which are different for the electron and for the electromagnetic fields.

Here we present a form to reconcile all the different formulations into a physically acceptable theory of the electron and the electromagnetic fields, including the known existence of **particle like** behavior: the electron and the photon.

We assume otherwise that the reader is familiarized with the Dirac equation and with Clifford algebras (remarking that a representation of the algebra of this paper is the well known algebra of the Dirac matrices, therefore no new algebra has been introduced, only a recognition that this algebra is the abstract form of the geometry of space-time-action).

4.1. Fields with Elementary Trajectories in START

We will now proceed to develop a theory for a field of carriers, embedded in the space-time-action geometry, with the simplest characteristic allowed by START. At the end we will show that the properties of these fields correspond to the experimental known properties of the electron, then within START we are developing a theory of the electron.

There are two types of fundamental trajectories in START, the ones defined by

$$dS^2 = 0 \quad ; \quad ds^2 = 0 \quad ; \quad da^2 = 0 \quad , \tag{70}$$

and the ones defined by

$$\begin{aligned} dS^2 \;&=\; 0 \quad ; \quad ds^2 \neq 0 \quad ; \quad da^2 \neq 0 \\ with \quad ds^2 - da^2 \;&=\; 0 \quad . \end{aligned} \tag{71}$$

The first one correspond to the cases of massless fields because the trajectories correspond to light-like paths and the energy momentum relationship $E = pc$, combined with the light-like trajectories ensure that $da^2 = 0$, that is

$$\begin{aligned} (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \;&=\; 0 \\ (Edt)^2 - ((p_x dx)^2 + (p_y dy)^2 + (p_z dz)^2) \;&=\; 0 \quad , \end{aligned} \tag{72}$$

whereas the second type of trajectories correspond to the particular case where, using $m_0 c^2 = E = h\nu$ and the particular choice of metric for the action coordinate $\kappa_0 = d_0/h = 1/m_0 c$ (where we have by definition a correspondence between d_0 and m_0), the relationship

$$(cdt)^2 - \left(\kappa_0 m_0 c^2 dt\right)^2 = (cdt)^2 - (da)^2 = 0 \quad . \tag{73}$$

Then either we construct a theory with the first type of fields only, that is all fields are in origin massless, or we construct directly a theory for a particular field where we have a rest mass m_0. The standard model of elementary particles has a mathematical structure corresponding to the first type of approach, all fields are massless before interaction . The common theory of the electron, otherwise has shown that for all properties which are not related to weak interactions, the direct use of the consideration of the electron as a carrier with mass m_0 is a useful approach. Here we will start with the second approach and in a second step show its extension to the first approach and then by necessity the inclusion of the electroweak interactions. We will follow this procedure because we will be analyzing different concepts in each case and because at the end there is no contradiction.

4.2. A Gauge Free Description of an at Rest Energy Distribution

According to our choice of description we consider a distribution of energy, $E(\mathbf{X}) = \rho(\mathbf{x})\bar{\varepsilon}$ such that $\partial E/\partial t = 0$, through our density of carriers $\rho(\mathbf{x})$ and an average energy per carrier $\bar{\varepsilon} = m_0c^2$. If the description corresponds to one carrier $m_0c^2 = h\nu_0$ with a proper frequency ν_0 , we then require that the density is normalized within a domain D

$$\int_D \rho(\mathbf{x})d\mathbf{x} = \mathbf{1} \quad . \tag{74}$$

The normalization corresponds then to the definition of all points of the distribution as corresponding to the field of the carrier. We mentioned above that the requirement of an equal energy per carrier corresponds to defining a given domain as the spacetime volume where we have a system of indistinguishable carriers. From the normalization itself it also follows that the proper frequency is not a property of the distribution but a property of the carrier which is being described. Likewise, for an observer in relative motion, a wave length will appear which is not a property of the distribution but a property of the carrier.

Of course this distribution is at rest only with respect to one frame of reference Σ with spacetime coordinates $\{x_\mu; \mu = 0, 1, 2, 3 \text{ and } x_0 = ct\}$. For any other frame of reference $\Sigma^{(0)}$ we must consider a Lorentz transformation R.

Associated to each matter field, there is an energy-momentum field $e_\mu p^\mu(x)$ (summation convention is used), denoting by $x = e_\mu x^\mu$ points in the Σ observers frame of reference, such that from the covariance of the energy momentum vector

$$e_\mu p^\mu(x) = m_0 c e_0' \quad , \tag{75}$$

assuming that there exists a (local) frame $e_\mu^{(0)}$ where the energy-momentum is the one corresponding to that of a carriers density at rest. The frame $e_\mu^{(0)}$ is related to the observers frame e_μ through the local Lorentz transformation

$$e_\mu^{(0)} = R(x) e_\mu R^{-1}(x), \quad R^{-1} = \tilde{R} \quad , \tag{76}$$

then (75) becomes

$$e_\mu p^\mu(x) = m_0 c R(x) e_0 R^{-1}(x) \quad , \tag{77}$$

we multiply (77) by $R(x)$ on the right

$$e_\mu p^\mu(x) R(x) = m_0 c R(x) e_0 \quad , \tag{78}$$

now, a crucial step in our program we give a definition to the field distribution of the carriers, here 1) the distribution is at rest in some frame of reference $\Sigma^{(0)}$ and for each point the spacetime trajectory is given by the vector $e_0^{(0)}$ and we will allow 2) for the field to have a possible spin $\mathbf{S}$ with plane of reference $e_1^{(0)} e_2^{(0)}$ (we should remain here that for any multivector M the Lorentz transformations is $R(x) M R^{-1}(x)$) for this purpose we now use the multivector double projector $P_{+\uparrow}$, with properties

$$P_{+\uparrow} = e_0 P_{+\uparrow} = P_{+\uparrow} e_0 \text{ and } P_{+\uparrow} = P_{+\uparrow} i e_1 e_2 \quad , \tag{79}$$

to obtain

$$e_\mu p^\mu R(x) P_{+\uparrow} = m_0 c R(x) P_{+\uparrow} i e_0 e_1 e_2 \quad . \tag{80}$$

Here the ie_1e_2 factor is to be kept for further reference to the fact that $P_{+\uparrow}$ was chosen as the appropriate projector, other choices could have been made. The up arrow refers to γ_{12} as the direction of spin up and the plus sign to the choice of "positive" mass m_0.

Now from our definitions of the gauge free representation of $E(\mathbf{X}) = \rho(\mathbf{x})\overline{\varepsilon}$, there is a function (a multivector Dirac spinor in fact as discussed in appendix B):

$$\psi(x) = A(x) R(x) P_{+\uparrow} \ \in STA \quad , \tag{81}$$

in the complex spacetime algebra, such that (80) can finally be written, through the use of the operator $p^\mu = i\hbar\partial^\mu$,

$$\hbar e_\mu \partial^\mu \psi(x) = m_0 c \psi(x) e_0 e_1 e_2 \quad , \tag{82}$$

where the i has been cancelled on both sides of (80). In the reference "rest" frame of the field $R(x) = 1$ and $A(x)$ should be such that

$$i\hbar e_\mu \partial^\mu A(x) = m_0 c A(x) e_0 e_1 e_2 \quad . \tag{83}$$

Our analysis here shows explicitly the multivector content of the Dirac spinor. The wave function (81) explicitly contains then 3 main contributions: the existence of the particles' field in $A(x)$, the relative motion of the particles' field in $R(x)$ and the reference to a preferred sign of m_0 and spin in $P_{+\uparrow}$.

This is a derivation from first principles of the, until now postulated only, Dirac-Hestenes equation [43, 44], this derivation is also an explanation of the geometric reason to consider a multivector equation which goes beyond the multivector analysis which was done (when wave function relativistic quantum mechanics was first developed) by solving the Dirac equation in terms of multivectors. The $\psi \in STA$ contains then a, local, Lorentz transformation and the information that a fixed time direction e_0 and a given plane e_1e_2 has been taken as an overall reference. But yet another element of information should be contained in $A(x)$; from the normalization consideration $\int_D \rho_0 d\mathbf{x} = 1$ the quantity $|A(x)|^2$ should have the dimensions of a density, $A(x)$ then contains a) a $\sqrt{\rho}$ factor, b) a gauge phase factor to allow for both interaction and freedom of description (discussed below) and c) a basic gauge factor f with the effect of the rest mass in (82) of the particle which should $f \to 1$ for a massless field.

Quantum mechanics is more general than (82), our analysis is in fact a starting point to reconstruct QM, given its basic reasoning.Here it is applied to the electron as elementary matter, spin $\frac{1}{2}$, fields, as a particular case, because the projector $P_{+\uparrow}$ and its eigenvector $ie_0e_1e_2$ was chosen as a reference. The general solution of (82), if $P_{+\uparrow}$ is not explicitly introduced in (82), will give a Dirac-Hestenes wave function ψ of the standard form mentioned above, discussed in more detail in appendix B, see (187) below, it contains four minimal ideals into one single wave function, but the amount of information is redundant as discussed in that appendix.

4.3. The Gauging of the Description of the Distribution

As mentioned above we can now proceed to the gauging of the auxiliary function introduced in (81) and the corresponding gauging of the equation (82). Here a very special situation arises because we can either use a scalar phase or in general any multivector phase compatible with (82). Because the operators have on both sides an odd number of vectors (either $e\mu$ or e_{012}) we can introduce a phase factor on both sides which has an even number of vector factors without any internal contradiction. That is the allowed phase factors are

$$e^{i\phi(\mathbf{X})} \quad \text{with} \quad \phi(\mathbf{X}) = \phi_{scalar}(\mathbf{X})\mathbf{1} + \phi_{pseudoscalar}(\mathbf{X}) i\gamma^5 + \phi_{\alpha\beta}(\mathbf{X})\gamma^{\alpha\beta}$$

The theory shows the reason for chirality being a basic property of nature as shown by the set of elementary particles. This can be clearly seen with the gauging of the Diracon equations

$$D_{(d,f)} = \Gamma^{\mu}_{(f)} \left[\partial^{(d)}_{\mu} - i \frac{e}{\hbar\hbar} A^{(d)}_{\mu}(\mathbf{X}) \right] \quad , \tag{84}$$

the gauging fields having the multivector composition (inducing chirality and local tetrads)

$$A^{(d)}_{\mu}(\mathbf{X}) = \tag{85}$$

$$= A^{d,scalar(electromagnetic)}_{\mu} + A^{d,pseudoscalar(weak,color)}_{\mu} i\gamma^5 + A^{tensor(gravity)}_{\alpha\beta,\mu} \gamma^{\alpha\beta}$$

that is, the gauging has electromagnetic, weak, color and gravity parts. The first two terms carry the index (d) because they are relative properties. Then the wave function becomes upon gauging (φ a reference spinor).

$$\psi_d(\mathbf{X}) = B \exp\left\{ i(p^{\mu}_d x_{\mu} + \phi_d(\mathbf{X})) \right\} \varphi \tag{86}$$

with the phase factor being a multivector

$$\phi_d(x) = \phi_{d,scalar}(\mathbf{X})\mathbf{1} + \phi_{d,pseudoscalar}(\mathbf{X}) i\gamma^5 + \phi_{d,\alpha\beta}(\mathbf{X})\gamma^{\alpha\beta} \quad , \tag{87}$$

the particular, relative, combinations for the phase, the $i\gamma^5$ terms, generate isospin and color and the $\gamma^{\alpha\beta}$ generate (as first shown by Fock and Iwanenko) the local Lorentz transformation which are a consequence of gravity.

4.4. The Consideration of the Electron Field as a Sum of Two Massless Fields

In the considerations of the previous sections the mass of the electron appears from the particular selection of a field where the density corresponds to a density of paths in the space-time-action geometry, using the unit of distance in the action direction to compensate for the unit of distance in the time direction in a simplectic (opposite signs) metric. This could be a fundamental definition in the geometry we are using as a frame of reference. Anyhow for comparison with the standard model of elementary particles, it seems preferable to start by considering massless fields, that is fields that in our geometry obey the second postulate because they have spacetime differential square $ds^2 = 0$ and action differential square $da^2 = 0$.

For this purpose we start by considering two massless fields L_0 and R_0 for spin $\frac{1}{2}$, then by necessity of fixed chirality the first being left handed and the second being right handed. A sum of these two fields will also obey the massless wave equation. For the construction of the massive field, introducing a gauging of the action through a gauge factor we will use now the following:

Theorem LK (Liu and Keller [85]**)** *There exists a suitable complex vector k_μ such that if $\Psi_0 \equiv L_0 + R_0$ satisfy the massless Dirac equation $i\gamma^\mu \partial_\mu \Psi_0 = 0$, then $\Psi = (\Psi_0)\exp(im \int k_\mu dx^\mu)$ will satisfy the massive Dirac equation $i\gamma^\mu \partial_\mu \Psi - m\Psi = 0$. Here*

$$k_\mu = k_\mu^+ + k_\mu^- = \frac{\pi_\mu^+}{2\overline{R_0}L_0} + \frac{\pi_\mu^-}{2\overline{L_0}R_0} \quad , \tag{88}$$

where $\pi_\mu^+ = \overline{R_0}\gamma_\mu R_0$; $\pi_\mu^- = \overline{L_0}\gamma_\mu L_0$; $\pi_\mu = \overline{R_0}\gamma_\mu R_0 + \overline{L_0}\gamma_\mu L_0 = \overline{\Psi_0}\gamma_\mu \Psi_0$.

The phase factor $\exp(im \int k_\mu dx^\mu)$ corresponds exactly to our definition in (74) of a carrier with one given energy mc^2 at every point of the distribution. The unit vector $k_\mu k^\mu = 1$ is needed to preserve relativistic covariance. The definition (88) shows the dependence on the mixing of the different handedness fields when Lorentz transformations are performed. For a field at rest with respect to the observer the only component of k will be $k_0 = 1$. We see then that even the mechanism for symmetry breaking implied in (88) is of purely geometric nature. Otherwise any field which could be coupled to either L_0 or R_0 will have to be acted by the same phase factor and then this mechanism for creating rest mass should be universal. This will be used in the next section in the description of the electroweak interactions, there when the weak interaction acts on the electron (positron) the corresponding carrier W^- (W^+) will carry away not only the charge of the field but also the coupling in the form of mass.

The fact that the particles acquire mass from geometrical considerations should be reflected in any combinations of fields which can be at rest with respect to an observer. Only the total number of contributions can change and then the masses of other elementary fields would be forced by the present theoretical considerations to be expressible as some algebraic function of the mass of the electron or at least through a mechanism similar to our LK theorem [73, 75, 78].

4.5. The Electroweak Interaction of the Electron Field

In order to understand the interactions of a field we have to consider first that the name itself expresses a relative property, the relation exists between to carrier fields which should have some gauge freedom of description which allows them to be considered together as expressed in one of the sections above. We saw the case of the electromagnetic interaction and in the description of the gauging of the electron field we already mentioned the possibility of considering bivector valued phases or pseudoscalar valued phases. The bivector valued phases were found long time ago, by Fock and Iwanenko (see appendix B), to correspond to gravitational interactions. Here we will describe the electroweak interaction as an extension of the electromagnetic case. We will show that it correspond to a pseudoscalar valued phase. For this purpose we first need to consider the partner of the electron in the weak interaction: the neutrino, next section, before analyzing the electroweak theory within our formalism.

4.5.1. The Theory of the Neutrino

We develop here a theory of the neutrino which is the natural complement of the formulation we have developed above for the electron.

We mentioned that we can define the electron field in an operational form: it is that field Ψ_e which obeys the Dirac equation (and its gauging)

$$(i\hbar D_0 - m)\Psi_e = 0 \quad ; \quad -(i\hbar D_0 - m)(-i\hbar D_0 - m) = -\hbar^2 D_0^2 - m^2 \,, \tag{89}$$

then the field has the correct mass, charge and spin density (of course magnetic and electric moment when gauged by the electromagnetic field).

The neutrino is considered to be massless, spin $\frac{1}{2}$ and uncharged, then without magnetic or electric moment and allows no gauging by the electromagnetic field. We propose then an operational definition: the neutrino field corresponds to that field which obeys the equation

$$D_n \Psi_\nu = 0 \quad ; \quad D_n D_n^\kappa = \partial^\mu \partial_\mu \quad , \tag{90}$$

with D_n such that the neutrino, besides being massless ($m = 0$ in (90)) is also neutral, no electromagnetic gauging (coupling) allowed and has the correct spin and chirality (left handed).

This properties are obtained from the definitions

$$\begin{aligned} D_n &= \gamma^0 \partial_0 + i\gamma^5 \gamma^j \partial_j \quad ; \quad j = 1, 2, 3 \\ D_n^\kappa &= \gamma^0 \partial_0 - i\gamma^5 \gamma^j \partial_j \quad , \end{aligned} \tag{91}$$

where we should remark that from the metric and the anticommuting properties of the basis vectors

$$-i\gamma^5 \gamma^i \partial_i i\gamma^5 \gamma^j \partial_j = -\partial^j \partial_j \quad , \tag{92}$$

$$i\gamma^5 \gamma^0 \partial_0 \gamma^j \partial_j - \gamma^j \partial_j i\gamma^5 \gamma^0 \partial_0 = i\gamma^5 \gamma^0 \gamma^j (\partial_0 \partial_j - \partial_j \partial_0) = 0 \quad . \tag{93}$$

Also, if a mass term $m \neq 0$ were included in (90) there will exist a spurious term $D_n m - m D_n^\kappa \neq 0$ which in fact prevents the use of $m \neq 0$.

Additionally from (90) and (91) the requirement $i\gamma^5\Psi_\nu = -\Psi_\nu$ for the neutrino imposes the condition for it to be a left handed particle field.

With respect to the gauging of the wave function Ψ_ν and the operator D_n, it is immediate that a term $\gamma^0 q_n A_0 g$ can not be cancelled by a gauge factor $g = e^{ia(t)/\hbar}$ acted upon by $i\hbar i\gamma^5\gamma^0\partial_0$ as far as $-\hbar\gamma^5\gamma^0\partial_0 g = -a'(t)\gamma^5\gamma^0 g$.

On the other hand a term $\gamma^5\gamma^0 A_0^{axial} g^{axial}$ will be cancelled by such a term. In the case of the electron fields both possibilities are open, then an axial electron current and an axial neutrino current can interact, this being the origin in our theory of the electron of the possible full electroweak interaction of the electron, but for the neutrino the interaction is restricted to the weak part (axial current only) and for the dual of the electron charge, that is g_D being the proper coupling constant.

4.5.2. The Electroweak Interaction of the Electron and the Neutrino

We then, from the specification of the neutrino as above, and the possibility of the electron left handed of right handed fields to be gauged by a pseudoscalar valued phase, can write a Lagrangian where both fields are together and the gauging of one corresponds to the opposite gauging of the other:

$$\nu_e + W^- \to e^- \quad \text{or} \quad e^- + W^+ \to \nu_e \quad , \tag{94}$$

with the new gauge field $W^-, W^+, [W^-, W^+] = Z^0$ carries four physical properties: charge, angular momentum, vector character and the possibility to interact with particles possessing a weak charge. Because this field interacts only with the left handed neutrino and the left handed part of the electron field, or the right handed antineutrino and the right handed part of the positron field, the weak field itself will have to interact with the mass producing phase factor of the LK theorem, then it will acquire mass from the same mechanism as the electron field (these matters are presented and analyzed in [58, 64, 67, 75, 78]).

4.6. Electrodynamics

For the electrodynamics of the electron within our formulation it is important to emphasize that we only have to particularize the analysis given above (Description in terms of interacting particles) to the case where the charge of the carrier $q = e$, as far as the rest of the analysis is general and doesn't need any change for the case of the electron.

4.7. The Theoretical Description of the Electron in START

It is convenient to summarized the resulting model for the electron when the description of a field with the basic properties of mass, spin and charge included. The action distribution is given a set of symmetries by requiring that the field corresponds to the most elementary field in START.

The average energy of the field is m_0c^2, because it obeys a wave equation in the START geometry the simplest representation corresponds to spin $\hbar/2$, its coupling to receive action from another field is given by e, the ratio e/m_0 corresponding to the rate of change of energy with action per unit energy of the original field. Correspondingly when action is given to other fields the strength of this action is also proportional to e. When work can be done on or by the field (redistributing the action), the emitted or absorbed energy per elementary action corresponds to a change of spin equal to $\hbar$, or to a corresponding change in angular momentum of orbital origin. We have to make a clear distinction between **actual** energies given to the field and **relative** energies which are described by a gauge field, even if in both cases the description of energy demands the definition of a frame of reference with respect to which one particle acquires more energy at the expense of the energy of the environment.

The elementary fields described by the model are required to be created or annihilated by at least pairs of fields with mutually canceling symmetry properties. Then a collection of fields, where no other fields are present which can cancel the symmetry properties of this collection, will correspond to stable matter. The distributions will exist in space but can only be created and annihilated in units of action , that is that the change in spin has to be a multiply of $\hbar/2$ and the change in energy with respect to a reference observer must correspond to $h\nu$. When fields at rest in a frame of reference are created (annihilated) the energy of the distribution is $h\nu = m_0c^2$. These conditions originate the notion of particle within the model, concept that will be even closer to the classical limit if the distribution domain is small compared to the distances involved in the global experimental observation. We can change the form of the action distribution in a continuous form but we can not change the existence of the distribution except in a (second) quantized form. At the same time it is now clear that at the level of the elementary fields (first) quantization corresponds to processes where the properties of the distributions are changed and second quantization to processes where distributions are created or annihilated.

4.7.1. A Massless Fields Approach to the Theory of the Electron

In the equations (90) and (91) above describing the neutrino we showed that a massless field with a given chirality is obtained as a necessary solution to the equation, that is in

$$D_n \Psi_\nu = 0 \quad ; \quad D_n D_n^\kappa = \partial^\mu \partial_\mu \quad ,$$

D_n was defined in such a way that the neutrino, besides being massless ($m = 0$) is neutral, then no electromagnetic gauging (coupling) is allowed and has the correct spin and chirality (left handed).This properties being obtained from the definitions in (91)

$$\begin{aligned} D_n &= i\gamma^5\gamma^0\partial_0 + \gamma^j\partial_j \quad ; \quad j = 1,2,3 \\ D_n^\kappa &= -i\gamma^5\gamma^0\partial_0 + \gamma^j\partial_j \quad , \end{aligned}$$

We can consider a similar equation for the massless fields of the electron

$$D_{me}\Psi_{me} = 0 \quad ; \quad D_{me}D_{me}^\kappa = \partial^\mu\partial_\mu \quad , \tag{95}$$

with D_{me} such that the fields for the electron, in a first stage are massless ($m = 0$), also neutral, as far as again no electromagnetic gauging (coupling) is allowed from the structure of the equations, and have the correct spin and chiralities (one solution can be left handed and another right handed). Properties obtained from the proper definitions

$$\begin{aligned} D_{me} &= i\gamma^5\gamma^\mu\partial_\mu \quad ; \quad \mu = 0,1,2,3 \\ D_{me}^\kappa &= -i\gamma^5\gamma^\mu\partial_\mu \quad . \end{aligned} \tag{96}$$

The general solution to (95) is a sum of a left handed field L and a right handed field R

$$\Psi_{me} = a_L L + a_R R \quad ; a_L^2 + a_R^2 = 1 \quad , \tag{97}$$

and from the LK Theorem above (88) we obtain the massive field Ψ_e. The general massless solution (97) allows no gauging except in the case where one of the coefficients is zero, where the gauging would proceed through an axial current, given the chiral properties of L and R.

For the weak interaction we now consider that the model shows the electron of being capable of coupling, including the interaction with a neutrino, from the set of action contributions below.

The Lagrangian in the standard model for a fermion field with electroweak interactions and a symmetry breaking mass term is reproduced from the considerations above ($L_0 = \psi^{(0)}_{e_L}$, $L_R = \psi^{(0)}_{e_R}$):

$$\begin{aligned}
\mathbf{L} \;=\; & +\frac{1}{2}\overline{L_0} i\gamma^\mu (\frac{1-i\gamma_5}{2})(\partial_\mu L_0 + \frac{ig'}{2} B_\mu L_0 - \frac{ig}{2} A^i_\mu \tau_i L_0) \\
& -\frac{1}{2}(\partial_\mu \overline{L_0} - \frac{ig'}{2} B_\mu \overline{L_0} + \frac{ig}{2}\overline{L_0}\tau_i A^i_\mu) i\gamma^\mu (\frac{1-i\gamma_5}{2}) L_0 \\
& +\frac{1}{2}\overline{R_0} i\gamma^\mu (\frac{1-i\gamma_5}{2})(\partial_\mu R_0 + ig' B_\mu R_0) - \frac{1}{2}(\partial_\mu \overline{R_0} \\
& -ig' B_\mu \overline{R_0}) i\gamma^\mu (\frac{1-i\gamma_5}{2}) R_0 \\
& -g_e \left[\overline{R_0}\Phi^\dagger (\frac{1-i\gamma_5}{2}) L_0 + \overline{L_0}(\frac{1-i\gamma_5}{2})\Phi R_0 \right] \quad ,
\end{aligned} \tag{98}$$

where the $\tau_i (i = 1, 2, 3)$ are Pauli matrices, Φ is a field corresponding to the LK Theorem, and B_μ and A^i_μ are $U(1)$ and $SU(2)$ gauge fields. A further analysis of the fields in (98) shows that the coupling constants g and g' correspond to the electromagnetic constant e and to its dual (axial) pair.

To (98) we should add the energy corresponding to the neutrino, the energy corresponding to the interactions fields and the possibility of the neutrino and the electron interacting via the axial current, which by definition is also the basic current of the neutrino from its chiral properties as a massless field. The neutrino, as mentioned above, can not interact with its polar current without violating spacetime symmetry.

4.8. The Local Structure of the Action Density

The basic postulate for the theory of matter and interactions fields here presented is the existence of a distribution of action, with certain geometrical characteristics, in a region of spacetime.

For massless fields a local system of vectors, at each point, defines the geometric characteristics: a vector a in direction e_5 for the intensity of action, a vector in direction e_0 for the, unavoidable, time direction. The rate of change of a with respect to time being the density of energy of the massless field $\varepsilon = \rho(X)h\nu$. The remaining three directions are internally defined in this case because all massless fields are chiral, including the interaction fields. One of the space-like directions corresponds to the direction of propagation of the massless field, with momentum $p = \rho(X)h/\lambda$ parallel to velocity. The remaining two space-like directions define the plane of the spin, perpendicular by definition to the propagation direction. This is the local structure of the action density corresponding to massless matter and interaction fields, this structure is by no means trivial and is reflected in the gauging properties of the description of the field. We should stress here that every point of the distribution is given the same set of local properties. The gauge freedom can only change the relative values of these sets of vectors at different spacetime points with the condition that the local structure has to be respected. Chirality and the relation $\varepsilon'/p' = c$ are basic features of the description of the massless fields, the primed energy momentum components related to the unprimed ones by a Lorentz transformation.

For massive fields the local structure is similar except that now the space-like set of three vectors, even if their directions are related to momentum and spin, offer now the additional gauge freedom of the direction of the plane of the spin being orientable independently of the momentum direction. In fact, for the electron there is at least one local frame of reference where $p = 0$ and, according to our principles above, $\partial a/\partial t = m_0 c^2$ also $(dx^5)^2 = (dx^0)^2$. For the massive fields, electron being our example here, we have then that for a general observer there is a current $\mathbf{j}$ which is related to the local frame where the particle field is at rest (frame vectors $\left\{\gamma_u^{(0)}\right\}$) through a Lorentz transformation

$$\mathbf{j} = j^\mu \gamma_\mu = \rho v^\mu \gamma_\mu = R\rho\gamma_0^{(0)} R^{-1} \quad , \tag{99}$$

showing that the distribution corresponds to a current $\mathbf{j}$ one of the basic properties of the matter field. The direction of the current and the direction of the momentum are two different quantities, the current related to the transformation of the local frame of reference and the momentum to the rate of change in space of the action distribution. In a similar form the spin $\mathbf{s}$ is related to a spin plane in the local frame of reference $\gamma_{12}^{(0)}$ by the same Lorentz transformation

$$\mathbf{s} = \rho s^{\mu\nu} \gamma_{\mu\nu} = R\rho\gamma_{12}^{(0)} R^{-1} \quad . \tag{100}$$

Idealized currents can be considered, for computational purposes, which can not be realized in nature. The best known example being perhaps that of a plane wave where in practice no real currents of matter or radiation can be approximated by such a current except for a very small region of space, and, moreover, only in the case where the actual current corresponds to a (steady) current of matter or radiation, consisting of a large number of matter or radiation units.

Because our description of matter and its interactions has as starting point a definition which corresponds to a definite picture of nature, there is a temptation to interpret several consequences of the theory as physical descriptions of nature even if in many cases they are only one form of description among many or if they correspond to an approximate description useful for calculations but not to a comprehensive description of the different phenomena. On the other extreme we can analyze the different feature of the theory in relation to well established mathematical models. Our theory reproduces the mathematical structure of density functional theory and the mathematical structure of quantum mechanics, then the calculational procedures can be carried on, having either of both presentations as a guide, given that the mathematical structure allows it. A basic difference is that we are using a continuous model for matter: a distribution of action where every point is endowed of a geometrical feature. The local geometrical feature, this mapping arising from the complex structure of space-time-action, corresponds to that of a Kerr spinning particle. The Kerr geometry itself can be considered as a special type of string, as has been discussed in length by Burinskii [12]. The main difference is that while Burinskii and other authors, since the pioneering paper of Carter [17], consider the Kerr geometry (a geometric structure reduced to a point) as the particle itself and then they are not using a continuous description of matter, our presentation starts with the distribution and only afterwards the local properties of the distribution can be recognized to correspond to the Kerr geometry, that is in our approach the distribution is of action where at each point a string-like structure, providing geometrical constrains to the action, exists with a weight $\rho(X)$.

5. Appendix A. Geometry to Complex Geometry

In mathematical physics it is useful and customary to use the concept of spacetime as a frame of reference for the description of the matter and their interaction fields. This corresponds to postulating a specific approach to geometry and to Geometrical Analysis. Spacetime, having a multivector structure and containing a spinor (and dual spinor) space, not only describes our perception of the physical nature but is also a powerful mathematical tool [64]. For our construction of a theory of the electron we adopt spacetime, with an inherent multivector structure provided by the Clifford ring $Cl_{1,3}$, as a basic frame of reference for physical phenomena, implying that its structure and symmetries correspond to the observed characteristics of the matter and interaction fields.

The abstract spacetime Clifford algebra $Cl_{1,3}$ is considered as generated by four basis vectors $\gamma_\mu, \mu = 0, 1, 2, 3$ with the basic property

$$\gamma_\mu \cdot \gamma_\nu = \frac{1}{2}\left(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu\right) = g_{\mu\nu} = \mathbf{diag}(1, -1, -1, -1) \quad . \tag{101}$$

The multivector algebra, algebra of geometric character, being generated by the totally antisymmetric (outer) product $\wedge$, where the bivectors representing the planes are the $\gamma_{\mu\nu}$

$$\gamma_{\mu\nu} = \gamma_\mu \wedge \gamma_\nu = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \quad , \tag{102}$$

and a basis for trivectors (sometimes called axial vectors), representing the volumes, $\gamma_{\mu\nu\lambda}(\mu \neq \nu \neq \lambda \neq \mu)$

$$\gamma_{\mu\nu\lambda} = \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\lambda \quad , \tag{103}$$

and the spacetime hypervolume $\gamma_{\mu\nu\lambda\rho} = \gamma_5$, (all μ, ν, λ and ρ different).

In complex spacetime a general multivector is a geometrical quantity $M = M^A\gamma_A$ with M^A a **real or complex** number and γ_A any of the 16 elements $\{1, \gamma_\mu, \gamma_{\mu\nu}, \gamma_{\mu\nu\lambda}, \gamma_5\}$ corresponding to the spacetime Clifford algebra. The index A runs over the 16 basic dimensionless elements of $Cl_{1,3}$.

In spacetime Clifford algebra the Clifford numbers are operators among themselves. In particular the bivectors $\gamma_{\mu\nu}$ are the generators of the Lorentz transformations:

$$M' = RM\tilde{R},\ M = M^A\gamma_A;\ R = \exp(\theta^{\mu\nu}\gamma_{\mu\nu});\ \tilde{R} = R^{-1} = \exp(-\theta^{\mu\nu}\gamma_{\mu\nu}). \tag{104}$$

The field of local Lorentz transformations $R(\mathbf{x})$ applied to the basis vectors γ_μ generate the Frenet tetrads e_μ related to a family of world lines $S(\tau)$ parametrized by τ, generally the proper time of a system, such that $e_0 = v$ the, local, velocity of the field. Then

$$e_\mu(\mathbf{x}) = R\gamma_\mu\tilde{R} = \Lambda_\mu^\nu(\mathbf{x})\gamma_\nu, \mathbf{x} = \mathbf{x}(\tau) or \theta^{\mu\nu} = \theta^{\mu\nu}(\mathbf{x}(\tau)) \quad . \tag{105}$$

Otherwise the geometric complexification

$$Cl_{1,3}^{(R)} \rightarrow Cl_{1,3}^{(C)} \quad \text{can be denoted by} \quad \left\{\gamma^A + i\gamma^A; \gamma^A \subset Cl_{1,3}^{(R)}\right\} \quad , \tag{106}$$

and real coordinates are to be used.

Multivectors act as operators among themselves and on the spinors ψ's, which are multivector ideals, describing the matter and interaction fields.

The best-known examples are γ_0, generating the parity inversion P; the trivector γ_{123}, generating the time inversion T; the bivectors γ_{0i}, generating the Lorentz boosts $\mathcal{L}$, the bivectors γ_{ij}, generating the space rotations $\mathcal{R}$; the tetravector γ_5, generating the duality transformation D; and the complex tetravector $i\gamma_5$, generating the chirality projection.

This combination of possibilities shows the immediate usefulness of using the space-time-action geometry as a basic frame of reference to construct the theory of the electron. We should also point out that the properties (101), (102) and (103) show that in G_{STA} the use of both commuting and Grassmann variables are an essential geometric ingredient (in fact of any geometry of dimension $N > 1$).

We now proceed to the definition of a geometry, in our sense a vectorial space which contains a carrier (generating) subspace. Define the calculus relevant to this multivector structure and consider the particular case of the isomorphism between complexification of the multivector fields and real multivector fields in a higher dimensional geometry.

5.1. Geometrical Analysis

Geometric analysis, the tool we are using in this paper, is a unification of subjects that is the natural extension of complex analysis (corresponding to a R^1 and Cl_1 and to a R^2 manifold and a Clifford algebra Cl_2^+ respectively) to the case of a R^n manifold and a Clifford algebra $Cl_{p,q}$ with $p + q = n$.

The program contains four mayor steps:

5.1.1. Selection of a $\mathbf{R^n}$ *Manifold*

corresponding to a quadratic space and the establishment of a set of n coordinates of this carrier space, collectively denoted by x. As a quadratic space there exists a map $(x, x) \rightarrow R$ in the form of a (local) real polynomial $P(x, x) = \sum_{ij}^n a_{ij} x_i x_j$. Where the coefficients $a_{ij} = (b_i^k)^\kappa b_j^l g_{kl}$, the b_j^l being elements of a quadratic space with a conjugation $(b)^\kappa$ such that $(b_i^k)^\kappa b_j^l$ is a real number. Our choice below $x \rightarrow \mathbf{x} = \mathbf{x}^{\boldsymbol{\mu}} \gamma_{\boldsymbol{\mu}}$.

5.1.2. Selection of a Clifford (then geometric) Algebra

$Cl_{p,q}$ with $p + q = n$, which automatically introduces the notion of geometric space and of a metric. The Clifford algebra $Cl_{p,q}$ is generated by the set of $n = p + q$ (anticommuting) elements $\left\{\gamma_\mu = \langle\gamma_\mu\rangle_1\right\}$, where $\langle\rangle_r$ denotes the rank (or number of factors) in the geometric product defined below, with $\mu = 1, 2, ..., n$, considered as basis vectors with a metric $g_{ij}\mathbf{1} = \frac{\mathbf{1}}{\mathbf{2}}(\gamma_i\gamma_j + \gamma_j\gamma_i)$, as defined below, and by definition $\gamma_i\gamma_j = -\gamma_j\gamma_i + 2g_{ij}\mathbf{1}$ denoting the unit element by $\mathbf{1}$. (For extensive descriptions of Clifford algebra and its relation to geometry [6, 43, 44, 46]). The 2^n–dimensional Clifford algebra is generated by, the repeated use of, the geometric product of any two basis elements (denoted by putting the elements together)

$$\gamma_i\gamma_j = \frac{1}{2}(\gamma_i\gamma_j + \gamma_i\gamma_j) + \frac{1}{2}(\gamma_i\gamma_j - \gamma_i\gamma_j) \quad , \tag{107}$$

the resulting product can be separated into the symmetric part of the product and the antisymmetric part of the product, it is customary to denote this operations dot and wedge parts of the product.

$$\gamma_i\gamma_j = \gamma_i \bullet \gamma_j + \gamma_i \wedge \gamma_j \doteq g_{ij}\mathbf{1}+\gamma_{ij} \quad . \tag{108}$$

Unfortunately the common nomenclature calls the process of selecting the symmetric part "dot product", and that of obtaining the antisymmetric part "wedge product" which is not basically correct but easy to remember.

An orthonormal set of basis vectors, standard choice which is not a limitation, is then one where if $\gamma_\mu, \gamma_\nu, \gamma_\lambda, .. \in \left\{\gamma_\mu = \langle\gamma_\mu\rangle_1\right\}$ the metric is diagonal with p values $+1$ and q values -1 or

$$g_{ij} = diag(1, ..., 1^{(p)}, -1, ..., -1^{(n)}) \quad , \tag{109}$$

A geometry is introduced into the manifold when the coordinates are embedded into the Clifford algebra

$$x \to \mathbf{x} = x^\mu \gamma_\mu \quad , \tag{110}$$

to obtain a 2^n dimensional **geometric space** (multilinear algebra) with the structural properties of the Clifford algebra, when $P(\mathbf{x}, \mathbf{x}) = \mathbf{x} \bullet \mathbf{x}.$

5.1.3. Construction of a Manifold in a More General Geometry than the Flat Basic Clifford (geometric) Algebra

by a series of functions and transformations generated by a mapping $\mathcal{M}$ of the manifold coordinates $x \to x' = f(x)$ where the coordinates x' contain now the particular characteristics of the description of the space. This procedure requires the study of the transformations S of the vectors and multivectors associated with $f(x)$. For any vector v we have the linear function $\mathbf{f}$ generated by $\mathcal{M}$

$$\mathbf{v} \to \mathbf{v}' = \mathbf{f}(\mathbf{v}) \quad . \tag{111}$$

by a suitable application of $\mathbf{f}$ to all vectors involved in the mapping $\mathcal{M}$ we generate, by outermorphism, the transformation of any multivector.

The function $\mathbf{f}$ is then a vector valued linear function of a vector. Because there are various forms of proceeding with the analysis we can define a set of four interrelated functions

$$\left\{\mathbf{f}, \mathbf{f}^{-1}, \mathbf{f_D}, \mathbf{f_D^{-1}}\right\} \quad , \tag{112}$$

which allow the free transit from x to x' in relation to the vectors and multivectors.

The inverse operation corresponds to the definition

$$\mathbf{f}(\mathbf{f}^{-1}(\mathbf{v})) = \mathbf{f}^{-1}(\mathbf{f}(\mathbf{v})) = \mathbf{v} \quad , \tag{113}$$

and the dual operations $\left\{\mathbf{f_D}, \mathbf{f_D^{-1}}\right\}$ allow the exchange of variables when both $\mathbf{f}(\mathbf{v})$ and v are expanded in relation to some particular vectors. By induction, applying the transformation to each vector, the set of functions are also defined for any multivector of $Cl_{p,q}$.

The procedure of this item allows the correspondence of a geometry $G_{p,q}$ associated to a manifold R^n to the manifold itself. Given that the multivector geometry $G_{p,q}$ has 2^n dimensions (degrees of freedom), corresponding to the 2^n elements of $Cl_{p,q}$, then the geometry $G_{p,q}$ contains an implicit set of 2^n fields $X^M(x)$ where M will take 2^n values to have a multivector field $X^M(x)e_M$ with the $\{e_M;\ M = 1, 2, ..., 2^n\}$ being a complete basis of the Clifford algebra.

5.1.4. Mappings and Transformations

Once we have the geometry $G_{p,q}$ and its basic functions we need to introduce the basic operators to transform the different fields and relate them among themselves: Transformations Operators,Differential Operators, Integral Operators, and Special Operators.

This will allow the study of the structure and symmetries both of the $G_{p,q}(R^n, Cl_{p,q})$ geometry and of the geometrical fields constructed over $G_{p,q}$.

We could of course consider the use of **complex fields** over $G_{p.q}$. This is different from the possibility of constructing a geometry from C^4 which is not necessary in practice. In that case we will embed the manifold into a $Cl_{p',q'}$, $p' + q' = n + 1$ Clifford algebra and the number of fields will change from $2^n \rightarrow 2 \times 2^n = 2^{n+1}$ as corresponds to a complexification of the full algebra, doubling the number of elements, with a faithful description of the new geometry. Complexification of a geometry corresponds to a change from R^n to a R^{n+1} manifold where the new degree of freedom is otherwise an induced coordinate with no special topology.

5.2. Derivation Operators

In the geometry which was discussed up to now one can extend the applied notation in order to include in a transparent form also other widely used notations. Below, we introduce the use of the dual frame γ^α which is defined as $\gamma^\alpha\gamma_\beta = \gamma_\beta\gamma^\alpha = \delta^\alpha_\beta$, and thus corresponds to a derivative ∂_{γ_α}, i.e., both $\gamma^\alpha\gamma_\beta = \delta^\alpha_\beta$ and therefore $\partial_{\gamma_\alpha}\gamma_\beta = \delta^\alpha_\beta$, can be equivalently used.

Consider for example the generator $\gamma_{\alpha\beta}$ of the rotation operator,

$$\widetilde{\mathbf{R}} = e^{\frac{1}{2}\theta(\alpha\beta)\gamma_{\alpha\beta}} \quad , \tag{114}$$

used to rotate a (multi-)vector $M = \left\{f^A(x)\gamma_A\right\}$, $\gamma_A \in \{1, \gamma_\mu, \dots .\}$,

$$M \to M' = \widetilde{\mathbf{R}} M \widetilde{\mathbf{R}}^{-1} \quad , \tag{115}$$

in the plane spanned by γ_α and γ_β. In eq. (115) $\widetilde{\mathbf{R}}$ leaves $f^A(x)$ unchanged, but transform the Clifford numbers γ_A, $\gamma_A \to \gamma'_A = \widetilde{\mathbf{R}}\gamma_A\widetilde{\mathbf{R}}^{-1}$. Alternatively, we can, as very often is the case, write the complementary transformations as

$$M \to M^{(r)} = (Rf^A(x))\gamma_A \quad , \tag{116}$$

where an equivalent rotation $f^A(x) \to (f^A(x))' = Rf^A(x)$ is used such that $\gamma_A \to \gamma_A$, and R is generated by

$$x^\beta\partial_{x^\alpha} - x^\alpha\partial_{x^\beta} \quad . \tag{117}$$

The (multivector-) algebra equivalent to eq. (117) implies a change of the frame of reference, namely

$$\gamma_\beta\partial_{\gamma_\alpha} - \gamma_\alpha\partial_{\gamma_\beta} \quad , \tag{118}$$

where $\partial_{\gamma_\mu} = \gamma^\mu$, and where in an orthonormal frame of reference $\gamma^\mu = g^{\mu\nu}\gamma_\nu$, with $g_{\mu\nu} = diag(1, -1, -1, -1)$. Since for a proper spacetime rotation we require $g^{\alpha\alpha} = g^{\beta\beta} = -1$, eq. (118) yields

$$\gamma_\beta\partial_{\gamma_\alpha} - \gamma_\alpha\partial_{\gamma_\beta} = 2g^{\alpha\lambda}\gamma_\beta\gamma_\lambda = 2g^{\alpha\lambda}\gamma_{\beta\lambda} = 2\gamma_{\alpha\beta} \quad , \tag{119}$$

which in turn shows the correspondence between eq. (118) and the generators of eq. (114). If for position vectors $\mathbf{x} = x^\alpha\gamma_\alpha$ eqs. (117) and (118) are applied simultaneously with angles θ and $-\theta$, they cancel each other.

A representation of the rotations of the Poincare group is therefore given by

$$R(\alpha\beta) = (x^\beta\partial_{x^\alpha} - x^\alpha\partial_{x^\beta}) + (\gamma_\beta\partial_{\gamma_\alpha} - \gamma_\alpha\partial_{\gamma_\beta}) = (x^\beta\partial_{x^\alpha} - x^\alpha\partial_{x^\beta}) + \gamma_{\alpha\beta} \, , \tag{120}$$

Other examples for operator including derivatives are constructed in a similar form [85].

5.3. The Steps to Build a Complex Space

We first illustrate the procedure in one and two dimensions, where the basic idea can be visualized, then we present the general case.

Consider a one dimensional geometry $G_{1,0}$ spanned by the vector $\mathbf{e}_1$ and the unit scalar $\mathbf{1}$, such that $\mathbf{e}_1^2 = \mathbf{1}$ is the $Cl_{1,0}$ Clifford ring $(\pm\mathbf{1}, \pm\mathbf{e_1})$ of that geometry. The usual complex plane is commonly represented as $z = x\mathbf{1}+\mathbf{iy}$, with the property $\mathbf{i^2} = -\mathbf{1}$, that is in practice the ring has been enlarged to $(\pm\mathbf{1}, \pm\mathbf{e_1}, \pm\mathbf{i})$ but, at the same time, restricted by convention to $(\pm\mathbf{1}, \pm\,\mathbf{i})$. The complex plane is otherwise isomorphic to $\mathbf{z} = x\mathbf{e_1}+y\mathbf{e_{n+1}} = x^{\mathbf{1}}\mathbf{e_1}+x^{\mathbf{n+1}}_{\mathbf{(1)}}\mathbf{e_{n+1}}$; or $x^1 = x$ and $x^{n+1}_{(1)} = y$ where, for a proper and faithful representation we require: $\mathbf{e}_1^2 = \mathbf{e}_{n+1}^2 = \mathbf{1}$ and $\mathbf{e}_1\mathbf{e}_{n+1} = \mathbf{e}_{1\ n+1} = -\mathbf{e}_{n+1}\mathbf{e}_1 = -\mathbf{e}_{n+1\ 1}$ to have the ring $Cl_{2,0}$ of $(\pm\mathbf{1}, \pm\,\mathbf{e_1}, \pm\mathbf{e_{n+1}}, \pm\mathbf{e_{1\ n+1}})$, where again $(\mathbf{e}_{1\ n+1})^2 = \mathbf{i_1^2} = \mathbf{e_1 e_{n+1} e_1 e_{n+1}} = -\mathbf{1}$. Hence the "even" part (even number of products) of the ring $Cl_{2,0}$ is $(\pm\mathbf{1}, \pm\,\mathbf{e_{1\ n+1}})$ and can be represented, using $\tan\theta_1 = \partial x^{n+1}/\partial x^1 = A_1$ by $\mathbf{z} \to \mathbf{z} = \mathbf{e_1 z}$ or the mappings:

$$\mathbf{p} = x^{\mathbf{1}}\mathbf{e_1}+x^{\mathbf{n+1}}\mathbf{e_{n+1}} \Rightarrow \mathbf{z^1} = \mathbf{e_1 p} \Rightarrow \mathbf{z^1} = x^{\mathbf{1}}\mathbf{1} + (x^{\mathbf{1}}\tan\theta_{\mathbf{1}})\mathbf{i_1} = x^{\mathbf{1}}(\mathbf{1}+A_{\mathbf{1}}\mathbf{i_1}).$$

The special situation arises for higher dimensional spaces $n \geq 2$ in which a full collection of "complex" planes are generated by the products $\mathbf{e}_i\mathbf{e}_{n+1}(i = 1, ..., n)$. In particular we have the collection of complex points $\left\{p_{(i)}, i = 1, ..., n\right\}$ with coordinates $(x^i, x^i A_i)$ where $A_i =^d \tan\theta_i$.

Then only one extra vector $\{\mathbf{e}_{n+1}; \mathbf{e}_{n+1}\mathbf{e}_i = -\mathbf{e}_i\mathbf{e}_{n+1},\ \text{all}\ i = 1, ..., n\}$ is needed to complexify the algebra. A collection of bivectors $\mathbf{i_i} =^{\mathbf{d}} \mathbf{e_i e_{n+1}}$ is generated.

In the auxiliary coordinate spanned by e_{n+1} there are two types of contributions: the one, arising from a reference value $x^{n+1}_{(0)}$ to $x^{n+1}_{(i)} = x^{n+1}_{(0)} + \tan(a_i)x^i$, called l below and the one related to the complexification of the $\mathbf{e}_{i\ n+1}$ planes given by $x^i A_i$. In the correspondence between the geometry and physics l is related to curvature and the A_i to the gauge fields like the electromagnetic (Kaluza-Klein or more general, string or superstring, theories [120]) case. All the degrees of freedom of $G_{p,q}$, $p+q = n$ are complexified, that is we also have complex bivectors, trivectors, etc. In particular we have the mapping in space-time of a vector $p = p^\mu \mathbf{e}_\mu$ such that $p^2 = p^\mu p_\mu = m^2$, with m a real number, to $p' = x^\mu i\gamma_5 e_\mu$ where $\mid p' \mid^2 = p^\mu p_\mu = m^2$ and $\mid A \mid^2 \equiv \frac{1}{2}(AA^* + A^*A)$, or to $p'' = x^\mu(\mathbf{1}\cos(\mathbf{n}+\mathbf{t}(\mu))\frac{\pi}{\mathbf{2}} + \mathbf{i}\gamma_5\sin(\mathbf{n}+\mathbf{t}(\mu))\frac{\pi}{\mathbf{2}})\mathbf{e}_\mu$ with n and $t(\mu)$ integers and obtain again $\mid p'' \mid^2 = m^2$, which in the study of the physical problems would correspond to the mixing of vector currents and axial vector currents, as in the theory of electroweak and color interactions. See below.

5.3.1. 4-D to 5-D

Mapping of a geometric complex n-dimensional space

$G(Cl_{p,q}, C^n; n = p + q)$ into a geometric real space of n+1 dimensions $G(Cl_{p+1,q}; R^{n+1}; n = p + q)$. We have defined above a real geometry for a basis R^n manifold through the definition of its local geometrical properties as those corresponding to a Clifford Algebra $Cl_{p,q}$ of signature $g_{ab} = diag(1, ..., 1_p, -1, ..., -1_q)$, $p + q = n$, containing 2^n elements. We then showed that its complexification, containing 2^{n+1} elements, can be represented by a R^{n+1} basis manifold which acquires a real geometry by the definition of its geometrical properties through a Clifford Algebra $Cl_{p+1,q}$. This is achieved by the creation of an n–dimensional metric tensor from the $(n+1)$–dimensional equivalent geometry:

$$g_{\mu\nu} \to g^c_{\mu\nu} = g_{\mu\nu} - \frac{g_{n+1,\mu} \cdot g_{n+1,\nu}}{g_{n+1,n+1}} = g^c_{\mu\nu}\left[1 - m^2(k_\mu k_\nu - \frac{e^2}{m^2}A^a_\mu A^a_\nu)\right] , \quad (121)$$

with $\mu, \nu = 1, 2, ..., n+1$,where the $g^c_{\mu\nu}$ are in practice n–dimensional because, by definition $g^c_{\mu,n+1} = 0$ and in particular $g^c_{n+1,n+1} = 0$. The last equality shows the splitting of the complex term into a basic part $k_\mu k_\nu$ and a relative part $A^a_\mu A^a_\nu$.

Relation (121) is obtained from a local decomposition of the space in a part tangent to the original R^n manifold and a perpendicular part, through the use of multivector projectors $P_\perp$ and $1 - P_\perp$. The geometry of $G\,(Cl_{p,q}; R^n; n = p + q)$ is not restricted to a flat space but can be any space which is n–dimensional and analytical and can then, from Campbell's theorem, be embedded in a R^{n+1} geometrical (flat) space.

For the complex formulation the coordinates are $x^\mu_c = x^\mu_R + ix^\mu_I$ and the real scalar product is defined $\frac{1}{2}(AB^* + A^*B)$.

For many applications it is necessary to generate a real line element which, with the decomposition into parallel and tangent to R^n parts is $dS^2 = ds_0^2 + (g_{n+1,\mu}dx^\mu)^2$, with $ds_0^2 = g^c_{\mu\nu}dx^\mu_R dx^\nu_R$ the line element of the n–dimensional real space (time), which, by definition has no $n + 1$ component. The additional "vector" appears as the action direction, creating the concept that the **physical** spacetime corresponds to the space–time–action space.

The concept of a formal geometry in carrier quadratic spaces, and its relation with the current use of Clifford algebra, has recently been formulated by Keller and Weinberger [80]. Here we have particularized to the specific case of the 5^{th} dimensional quadratic carrier space, generated by the geometrical unification of space, time and action.

6. Appendix B. On our Knowledge of the Electron and the Mathematical Description of its Properties

The experiment of Gauss and Weber in the 1830's on the Ampere law led to conceive the electron and the atomic cores before the confirmation by J. J. Thomson in 1897 as the carrier of negative charge and the first studies of these particles determined it as a (point) charge with well defined trajectories and with a well defined charge to mass ratio. The charge $q = -1.602 \times 10^{-19}$ Coulomb and the mass $m_o = 9.108 \times 10^{-31}$ kg. Then the electron was to be defined as a massive charged particle, point-like up to the experimental accuracy of that time. In fact in the years 1846-1856 Weber developed a relative velocity dependent generalization of the Ampere's law that led him in 1871 to the theoretical recognition of the existence of the charged atomic nucleus and oppositely charged orbiting electrons. In 1855 Weber determined the constant velocity, $= \sqrt{2}\,c$ to be known as the Weber constant. Bernard Riemann observing the Weber experiment noted that the value of $\sqrt{2}\,c$ is close to the velocity of light determined by Fizeau.

Weber (1871) [118] arrives at the charge to mass ratio and proton-electron mass ratio, derived the formula $\frac{e^2}{mc^2}$ known as the classical electron radius and identified the nuclear binding force for which there was no empirical evidence until the 20th century. Weber also pointed in 1871 that his constant velocity $\sqrt{2}\,c$ *must represent a limiting velocity for the electrical particles.* This early history of atomic science was presented by Hecht (1996) [42]. The name *electron* was introduced in 1874 by G. Johnstone Stoney one year after the publication of the *Treatise on Electricity and Magnetism* by James Clerk Maxwell in 1873.

Let e and e' denote the charges m and m' be the masses. Weber denotes by f the acceleration not caused by mutual action of particles. Then the Weber law (1871) for the force between two charged particles in relative motion led him to the following comment:

From this its results that the law of electrical force is by no means so simple as we expect for a fundamental law to be.... The particles do not by any means always repel each other; (relativity theory cleared up these relations 35 years afterwards).

These considerations were soon confronted with a basic problem: a charged particle generates an electrostatic field of intensity $E(r)$ which contains an electromagnetic static field energy

$$U_{em} = \int_a^\infty \frac{1}{2k}(\mathbf{E}^2 + c\mathbf{H}^2)4\pi r^2 dr \quad \text{with} \quad \mathbf{E}(r) = \frac{kq}{r^2} \quad , \tag{122}$$

where a is the radius of an spherical distribution of the charge q, that is (without magnetic field contributions)

$$U_{em}(a) = kq^2/2a \quad . \tag{123}$$

It is well known then that, as for a point particle $a \to 0$, the energy of the electromagnetic field diverges. Moreover the self interaction of the field with the particle could also be considered $U_{self}(a) = +kq^2/a$ as an additional divergent quantity. When in the 20th century the mass-energy relation was discovered, it was thought that $U_{em}(a)$ could be the origin of the electron's mass

$$m_a c^2 = U_{em}(a) \quad , \tag{124}$$

which defines a quantity $r_o = 2a$, and if $m_a = m_o$ then

$$r_o = \frac{q^2}{m_o c^2 4\pi\epsilon_o} \quad , \tag{125}$$

These models replace the electron's mass by the electron's radius parameter. It is also well known that further refinements give for the spherical distribution of charge moving with velocity v

$$\epsilon_{em} = \gamma \frac{kq^2}{2a}\left(1 + \frac{1}{3}\frac{v^2}{c^2}\right) \quad , \tag{126}$$

$$P_{em} = \gamma \frac{2kq^2}{3ac^2} v \quad , \tag{127}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$, for the electromagnetic energy ϵ_{em} and the electromagnetic moment P_{em}. A discussion of these problems (Poincaré 1906, Abraham and Lorentz 1905-1909, Cohen and Mustafa 1986, Lozada 1989, etc.) is still a live subject in the literature. No structure or size has been found for the electron this far, the distribution related to it can be made as small in volume as available energy allows.

The picture of the electron was more complete when it was discovered that besides its current j_μ and its associated mass m_o , another property, the electron's magnetic moment μ_e , had to be introduced (Stern and Gerlach 1921). This vectorial quantity in the presence of an external magnetic field $\mathbf{H} = h\vec{e}_3$ can, have only two values $\mu_s = \pm e\hbar/2m_o\vec{e}_3$ (notice the change of units $q \to e$). The well known ideas of Pauli (1921) and of Goudsmit and Uhlenbeck (1925) conveyed to the introduction of the electron's intrinsic angular momenta, or spin, $\vec{s} = \hbar/2\vec{e}_3$.

Then already in 1925 the electron was to be considered as a massive charged particle with spin s and magnetic moment μ_e. A few years later quantum mechanics was introduced and it was found that the theory of Dirac (1928) gives a correct description of the electron in the sense that it provides useful calculational procedures for the electron in the one particle approximation. Of course it was a great success that the equation also described the positron. The latest definition of electron could be then operational: *An electron is a particle obeying Dirac's equation with charge* $-e$ *and mass* m_o*. The same equation introduces both the spin and the correct magnetic moment of the electron as a structural consequence.*

Users of the Dirac equation can work with many electron, fermion systems, if the self-Coulomb problem is avoided. Frequently this is done by solving at the same time the statistics problem through the systematic consideration of exchange and the Pauli exclusion principle, a common practice between atomic, molecular and solid state physicists.

In this paper we will, on the other hand, not separate the problem of the electron and its electromagnetic fields in order to search for new understanding of the electron's nature. We will find that the natural mathematical formulation is given in terms of multivector algebra.

We could otherwise try to avoid the explicit reference to the electromagnetic fields, (see references in **ET1** for Schwarzschild (1903), Tetrode (1922), Fokker (1929, 1932), Wheeler and Feynman (1949) or Sutherland (1989)).

6.1. Geometry and the Electron

Here we analyze the different possibilities to study the electron, in the single particle approximation, mainly from the point of view of the spacetime multivector Clifford algebra. Different points of view of the electron and their relation to Clifford algebras [20, 21, 22, 25] are confronted.

We further assume that any physical quantity in the spacetime should correspond either to a scalar (for example a charge), a vector (the electron current or the energy-momentum which, in principle, is a time component of a tensor, or the electromagnetic potentials A^μ from which the electromagnetic magnetic field intensities can be derived by, outer, differentiation), bivectors (as the electromagnetic field intensities), trivectors (the axial currents) or pseudoscalars (the spacetime hypervolume itself or quantities related to chirality). We have reminded the reader of these examples to illustrate that the spacetime Clifford algebra is already the basis for the study of the electromagnetic field or of the properties usually associated with the electron matter field itself. It has been shown that specific combinations of multivector quantities are associated to spinors ψ_M, see for example Hestenes (1979) or Boudet (1985). These multivectors being considered to belong specifically to the electron matter field, [2, 6, 8, 9, 16, 18, 19, 20].

The projection of those multivectors ψ_M corresponding to ideals or Hestenes spinors by right multiplication by a constant (unit) minimal ideal η_i (inverse Cartan map), results into the well known Dirac spinors ψ . We have shown [63] that the particular choice $\eta_{M,i}$ of the constant minimal ideals, where each i corresponds to a combination of signs in $(1 \pm \gamma_0)(1 \pm i\gamma_{12})$:

$$\eta_{M,i} = \frac{1}{4}(1\pm\gamma_o)(1\pm i\gamma_{12}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ etc} \quad ,(128)$$

or the "column" spinor equivalent η_i, for example for $i = 1$

$$\eta_i = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \eta_i^T = (1\,0\,0\,0) \text{ such that } \eta_{M,i} = \eta_i \eta_i^T \quad , \qquad (129)$$

results into the reconstruction of the multivector ψ_M corresponding to a spinor set $\{\psi_i\}$

$$\psi_M = \sum_i \psi_i \eta_i^T \quad , \qquad (130)$$

and the standard Dirac theory can be faithfully mapped, Cartan map, to a multivector, Dirac-Hestenes, theory.

The multivector representing the electron obeys the Dirac wave equation (eigenvalue equation). All results related to that formulation are kept in the new formulation: orthogonality, interference, dispersion, diffraction, etc. Moreover the statistical nature of the interpretation of the theory is enriched in the sense that we will be working at two "statistical" levels:

1) The wave nature of the fields of multivectors and their sum resulting in interference and

2) The distribution nature of the different multivector fields which in the standard theory, and more widely used interpretation, are taken to be the "probability density" of finding the particle at a given point, ρ, the probability density of finding the particle with a current j or a spin s, etc. We should not forget that the theory for a collection of particles should be treated within

many body quantum theory with its particular set of premises and rules, densities being replaced by density matrices where the fermion statistics are fully included.

A set of rules for the formalism in terms of multivectors and for the interpretation of the results computed with the theory allows the unambiguous calculation of the quantities obtained with the standard procedures. Nevertheless a richer physical structure of the theory expressed in terms of multivectors will be shown to exist, in particular, in relation to our basic claim that the matter and interaction fields are both parts of one single physical reality.

Electromagnetism has been known for a long time, Mercier 1934 [88, 89], to be a theory that can be constructed entirely on multivectors. The now well known discussions of Hestenes (1966) [41, 43, 44, 46] or Casanova (1976) [18, 19] illustrate its structure. The matter fields enter into the theory as charge-current distribution densities $j_\mu \to (\rho, \mathbf{J})$, (see equations (130) and (131) below). Nevertheless as the mapping multivectors $\to$ spinors is possible and the electromagnetic field needs a maximum of 6 quantities to be defined (spinors allow the use of 6 independent parameters), electromagnetism could also be considered a theory constructed through the use of Dirac spinors [60, 61, 62]. Here, mainly to emphasize our point of view of matter fields as distribution of multivectors, we will take the two possible formulations: the theory to be cast in terms of multivectors and in their multivector ideals. See below, section VI, for the particular construction $\mathbf{F} = \psi\gamma_{12}\psi*$.

In brief, for our analysis leading to the ideas below, we are either using the inverse Cartan map (Crawford 1985, see also [65]) or we use the Cartan map to analyze the multivector theories for matter fields.

The currents $\mathbf{J}$ (for example the generated by the electron field) can in general be decomposed in their solenoidal j_{sol} and irrotational j parts. In Dirac's theory the solenoidal parts contain two components: one which is intrinsically solenoidal and a second which is solenoidal only in reference to the boundary conditions and the observer's frame of reference. Then the electron sources of the electromagnetic fields are described in fact by a set of seven basic quantities:

$$\rho, j_i, j_j, j_k, j_{i,sol}, j_{j,sol} \quad \text{and} \quad j_{k,sol} \quad . \tag{131}$$

We already reminded the reader that an electron cannot exist without its electromagnetic fields, that is it exists with an electrostatic field, generated by the electron's charge, an **intrinsic magnetic field** generated by its intrinsic **solenoidal current** and an additional magnetic field generated by the, extrinsic, electric current. We should remember that the intrinsic solenoidal current is explicit when the Dirac's current is analyzed via the Gordon decomposition, Ohanian 1986 [92]. A satisfactory theory for the electron which is also a satisfactory functional definition of the electron should be obtained when the sources and the interaction fields are considered as a unit. Such a theory would be in accordance with a philosophical point of view that a physical entity is constituted by whatever is observable of it (an observation being understood as all that we can infer through experiment). The intrinsic solenoidal current of the electron implicates (see the detailed discussion of Ohanian (1986)) not only a magnetic moment but also an angular momentum

$$S = \int S(x) = \hbar/2 \quad , \tag{132}$$

then in (131) above j_{sol} could also be replaced by an angular moment field $\vec{S}(x)$. Dirac's theory shows that the magnitude of $\vec{S}(x)$ is

$$S(x) = S\rho(x) \quad , \tag{133}$$

then only the direction of $\vec{S}(x)$ is independent of $\rho(x)$ but not its magnitude, this is one of the most important features of the geometrical content of the electron theory. It says that, even if the analysis of an electron distribution shows some solenoidal current, there is a rotational of the distribution at every point and, as is well known in vector analysis, the overall intrinsic solenoidal current is the result of the application of the Gauss theorem to the ensemble. Then an analysis of the spin as resulting from a macroscopic current is mathematically correct but physically misleading: *every point of the distribution contains the same amount of angular momentum per unit density.*

There is no indication whatsoever of an structure giving rise to spin and in fact a spin field $\mathcal{S} = \psi\gamma_{12}\psi^* = \rho\psi\gamma_{12}\psi^{-1}$ is one of the most fundamental quantities of the theory

In all experiments performed up to date an electron appears as a distribution of charge, currents and electromagnetic (electroweak, in fact) fields. Most problems arise from the attempt to rationalize the experimental facts starting from a point particle idea as the basis for the interpretation of experiment or for the interpretation of the results of the now standard quantum mechanical calculations. Experiment shows that there is no internal structure of the electron, but the experiment does not disagree with the existence of distribution. The "interpretation" of the distribution is a fundamental question of quantum mechanics, not of the electron theory. That is, there is no experiment resolving the electron "cloud" into instantaneous positions of a "point" particle, nor, at the same time is there any evidence at all of a possible excitation of internal structures of an electron.

We could speak in terms of electromagnetic quantities alone. The densities which we commonly refer to the sources can be substituted by electromagnetic quantities through the integral form of the Maxwell equations. For example to relate $\mathbf{E}$ and $\nabla \cdot \mathbf{E}$

$$\mathbf{E}(\vec{r}_2) = \frac{1}{4\pi\epsilon_o} \int \frac{\epsilon_o \nabla \cdot \mathbf{E}(\vec{r}_1)}{r_{12}^2}\, \vec{r}_{12} dV_1 \quad , \tag{134}$$

or to relate $\nabla \times \mathbf{H}$ and $\nabla \cdot \mathbf{E}$ for time independent $\mathbf{E}$

$$\nabla \times \mathbf{H} = (\nabla \cdot \mathbf{E})\mathbf{v} \quad , \tag{135}$$

and we can even think of the electromagnetic potentials A^μ as quantities related to the sources in special forms

$$\nabla^2 A^o + \nabla \cdot \frac{\partial \vec{A}}{\partial t} = -\frac{\nabla \cdot \mathbf{D}}{\epsilon_o} \quad . \tag{136}$$

We can then assume that besides the field intensities $\mathbf{E}$ and $\mathbf{H}$ we have a vector distribution

$$\nabla \cdot \mathbf{E} \longrightarrow \rho \quad , \tag{137}$$

$$(\nabla \cdot \mathbf{E})\mathbf{v} \longrightarrow \mathbf{J} \quad , \tag{138}$$

and the energy momentum related to this vector being

$$\mathcal{E} = \gamma m_o \epsilon_o \nabla \cdot \mathbf{E} \quad , \tag{139}$$

$$P = \gamma m_o \epsilon_o (\nabla \cdot \mathbf{E})\, \mathbf{v}' \quad . \tag{140}$$

Here m_o appears as a parameter providing the correct dimensions and $\mathbf{v}'$ corresponds to the relative velocity between the inertial system where $\nabla \cdot \mathbf{E}$ has been computed and that of the observer. In equation (127) the quantity γ is to be computed from $\mathbf{v}'$. Remember that relativistically $\mathbf{E}$ and $\mathbf{H}$ can not be separated!, nor have they a unique formulation, in fact they can always be expressed as Lorentz transformations and duality rotation of a reference bivector $\mathbf{H} = \psi\gamma_{12}\psi^*$. See below.

The energy momentum of the electron will be given by (139) and (140) when $\mathbf{E}$, $\mathbf{H}$ and $\nabla \cdot \mathbf{E}$ are those (additive) quantities referred to the particular electron under consideration. $\mathbf{E}$ and $\mathbf{H}$ are to be used only in relation to other particles. Equations (137,138,139,140) are nevertheless insufficient.

6.2. Considerations About the Relationship Between the Vectorial Part of the Electromagnetic Field and the Dirac Spinor

We briefly reproduce here what was presented elsewhere (Keller and Viniegra 1992). The direct and inverse Cartan maps between Dirac spinors and spacetime multivectors, a procedure which incorporates Fierz identities and the Boudet (1985) relations, have allowed us to show the explicit relations. Here $\bar{\psi} = \psi^{+}\gamma_o$, and $\rho_A = \bar{\psi}\gamma_A\psi$, where the γ_A are here any of the 16 hermitian multivectors of the complex Dirac algebra, such that the corresponding real numbers are

$$\begin{array}{c} \sigma = \bar{\psi}\psi, \quad \pi = \bar{\psi}\gamma_5\psi, \quad \Sigma_{\mu\nu} = \bar{\psi}\gamma_{\mu\nu}\psi, \\ j_\mu = \bar{\psi}\gamma_\mu\psi \quad \text{and} \quad k_\mu = \bar{\psi}_1\gamma_\mu\gamma_5\psi. \end{array} \quad , \tag{141}$$

related through the Fierz identities

$$j_\mu j^\mu = \sigma^2 + \pi^2 \quad , \tag{142}$$

$$k_\mu k^\mu = j_\mu j^\mu \quad , \tag{143}$$

$$j_\mu k^\mu = 0 \quad , \tag{144}$$

$$\Sigma_{\mu\nu} = (\sigma^2 + \pi^2)^{-1} \left\{ \sigma\varepsilon_{\mu\nu\rho\tau} j^\rho k^\tau - \pi(j_\mu k_\nu - j_\nu k_\mu) \right\} \quad , \tag{145}$$

(here $M_\mu = M^\nu g_{\mu\nu}$). Through the use of the Crawford's inversion theorem (Crawford 1985), the ψ_i mentioned above can be constructed

$$\psi_i = e^{-i\phi_i}\rho_A\gamma^A\eta_i \quad , \tag{146}$$

where ϕ_i is an arbitrary phase and η_i is a reference constant spinor, which projects the multivector $M = \rho_A\gamma^A$ into the space of the Dirac spinors. This multivector M is the one which corresponds to the additional vectors and other quantities in the extended matter-interaction field theory proposed above. The multivectors obey the Dirac like equation (remember the η_i are constant)

$$\left[(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu)\psi_i\right]\eta_i^T = m_o c\psi_i\eta_i^T \quad , \tag{147}$$

where $\psi_i\eta_i^T$ is a multivector and the operator $(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu)$ is a spacetime vector operator. $\psi = \sum_i \psi_i\eta_i^T$ obeys the Hestenes equation for a particular set of values of the ψ_i. The relative phases ϕ_i have to be given in such a form that $M =< e^{\beta\gamma^5} m >_0 = m\cos\beta$.

6.3. Spin, de Broglie Waves and Mass

Let us compute the spin of a de Broglie wave (de Broglie 1968) of the form proposed by Mackinnon (1981)

$$\psi = f(r)g(\underline{x},t)\eta_1, P_{+\uparrow}\eta_1 = \eta_1 \quad . \tag{148}$$

Mackinnon proposed a particular case with $f(r) = \sin(kr)/kr$ but $f(r)$ can be in general a spherical Bessel (Newman) function. The momentum $\mathbb{P}$ of the field is

$$\mathbb{P} = \frac{\hbar}{4i}\left[\psi^\dagger\nabla\psi + \psi^\dagger\alpha(\alpha^\mu\nabla_\mu)\psi\right] + h.c. \quad , \tag{149}$$

$$\mathbb{P} = \frac{\hbar}{2i}\left[\psi^\dagger\nabla\psi - (\nabla\psi^\dagger)\psi\right] + \frac{\hbar}{4}\nabla\times(\psi^\dagger\boldsymbol{\sigma}\psi) \quad , \tag{150}$$

and, for $g(\underline{x},t)$ corresponding to a particle at rest

$$\mathbb{P} = \frac{\hbar}{4}\left(\frac{k^3}{2\pi^2}\right)\frac{\partial}{\partial r}\left(\frac{sin^2 kr}{k^2r^2}\right)(-2y\hat{x} + 2x\hat{y}) \quad , \tag{151}$$

which represents a circular flow of the field in the plane xcy. The circulation has a singularity (zero density weight nevertheless) at the origin. The angular momentum is given by

$$\mathbb{J} = \frac{\hbar}{2i}\int x\times[\psi^\dagger\nabla\psi - (\nabla\psi^\dagger)\psi]d^3x + \frac{\hbar}{2}\int\psi^\dagger\boldsymbol{\sigma}\psi d^3x \quad , \tag{152}$$

and again the second term, spin, will be the relevant quantity. If we assume f to be normalized then the integral of the spin part would be trivially of magnitude $\hbar/2$. As for a de Broglie wave packet should be $k = m_o c/\hbar$, then the same prefactor $f(r)$ that generates the mass generates the spin of the total wave. This seems to be the real origin of the structural parts discussed above. Notice that $4\pi r^2(f(r))^2 = 0$ as $r \to 0$.

A different, probabilistic, problem is related to the interpretation of the total ψ as a probability amplitude. As mentioned above each point in the space where the electron is considered is given the same factor m_o and $s = \hbar/2$ for mass and spin magnitude.

The interpretation of $f(r)$ is consistent with the discovery that zitterbewegung is mapped with a circular (helical) current for an electron at rest (in displacement). The classical pictures generated by spacetime algebra act as "localizer" of currents. This is seen both for bound states (see the discussion of the atomic calculation as plane solution in the book of Casanova (1976)) or for the traveling waves (Hestenes 1991, Vaz and Rodriguez 1993) as zitterbewegung.

The prefactor $f(r)$ provides, additionally, a connection with the standard model of elementary particles as far as

$$f(r) = \frac{\sin kr}{kr} = \frac{e^{ikr}}{ikr} - \frac{e^{-ikr}}{ikr} \quad , \tag{153}$$

and it corresponds to a standing spherical wave. e^{ikr}/kr is an outgoing spherical wave and e^{-ikr}/kr an incoming spherical wave. Given a spin direction they will have opposite helicities and the standing spherical wave will be the realization of the well known sum of a left-handed and a right-handed wave. The $\pm i$ factor in the exponent would be the eigenvalue of the γ_5 operator. This relation to the standard theory will be discussed in part II of this paper.

The formation of non-dispersive waves for the relativistic wave equation (Maxwell) started at least with the work of Bateman (1915). But one should not be confused with the fact that the solution Φ of the $\Box^2\Phi = 0$ may be given with Φ belonging to any of the $\mathcal{D}^{m,n}$ representations of the Lorentz group.

6.4. The 1929 Work of Fock and Iwanenko

As we have mentioned before (Keller, [68] comment 1992), the geometric, Clifford algebraic, content of the Dirac equation and the corresponding electron theory was clearly and explicitly discussed by V. Fock and D. Iwanenko in 1929 [32, 33, 34, 35]. The 20th of May 1929 a physics conference took place in Charkow (Ucrania) where Fock and Iwanenko presented the idea that the Dirac matrices had a pure geometrical meaning. The theory was developed in a series of 4 papers:

(A) V. Fock and D. Iwanenko, Über eine mögliche geometrische Deutung der relativistischen Quantentheorie, Z. für Physik **54**, 798 (1929).

(B) V. Fock and D. Iwanenko, Géométrie Quantique Linéaire et Deplacement paralléle (presented by Maurice de Broglie), C.R. Acad. Sciences (Paris) **188**, 1470, (1929),

(C) V. Fock, Geometrisierung der Diracschen Theorie des Electrons, Z. für Physik **55**, 261 (1929) and

(D) V. Fock, Sur les Équations de Dirac dans la Théorie de Relativité générale, C.R. Acad. Sciences (Paris), **189**, 25 (1929).

In paper (C) the Dirac equations were given a general relativistic invariant form and the gauge invariance of general relativity and electromagnetism were recognized as having a geometrical meaning by themselves and a proper place in the geometric description of the physical world: gravitation through the Ricci coefficients and electromagnetism through, an independent geometrical quantity, the four potentials. In papers (A) and (B) they explicitly wrote

$$d\mathbf{s} = \sum_{\nu} \gamma^{\nu} dx_{\nu} \quad , \tag{154}$$

as a linear differential form whose square $d\mathbf{s}^2$ should be equal to the Riemann ordinary quantity. They proposed to call this point of view of geometry: **linear quantum geometry** They also proposed that the Dirac spinor should be considered as a new geometric quantity (called semi-vector, following L.D. Landau). In short they proposed that the purely geometric quantities should be the Dirac algebra set of elements and to consider them as operators. This allowed Fock and Iwanenko to consider all geometric transformations related to special and general relativity. Having found the geometric meaning of the Dirac γ_A matrices they had no problem to formulate the theory in general coordinate system.

In the first section of their "Comptes Rendus Hebdomedaires des Sceances de l'Academie des Sciences"(1929B) paper these Rusian physicists state that in Riemannian geometry the fundamental quadratic form ds^2 explains gravitation but for quantum and electromagnetic phenomena new geometric notions are needed ("...notions geometriques nouvelles et ètrangères à la geometrie de Riemann"). They propose:

"...Le caractère géometrique des operateurs α_k de Dirac a été signalé par les auteurs de cette Note [Paper (1929 A) Ueber eine mögliche geometrische Deutung der relativistischen Quantentheorie] qui ont proposé d'introduire les operateurs analogues aux matrices de Dirac dans la géométrie et de considérer la forme différentielle lineaire

$$ds = \sum_{\nu} \gamma^{\nu} ds_{\nu} \quad , \tag{155}$$

dont le carre donne le ds^2 ordinaire de Riemann"...

There is little doubt that this is the birth of the spacetime geometry and geometric Clifford algebra concepts.

In the second section of (1929 B) they state that from the $n - tuple\, \gamma^{\nu}$ of orthogonal directions at each point of space we can define the Dirac spinor ψ, which they call <demi-vecteur>. Then, in analogy to the parallel displacement of a vector studied by Levy-Civita, the parallel displacement of a semi vector can be used to start linear geometry

$$\delta\Psi = \sum_{\nu} c_\nu ds_\nu \Psi \quad , \tag{156}$$

where the c_ν are now operators (matrices) on the Φ and the ds_ν the infinitesimal displacements defined in (154)

$$\delta\Psi^\dagger = \Psi^\dagger \sum_{\nu} c_\nu^\dagger ds_\nu \quad , \tag{157}$$

and if $\alpha_\nu\alpha_\mu + \alpha_\mu\alpha_\nu = 2\delta_{\mu\nu}$ and the $A_\nu = \Psi^\dagger\alpha_\nu\Psi$ behave as vector components, then

$$\delta A_\nu = \delta(\Psi^\dagger\alpha_\nu\Psi) = \Psi^\dagger \sum_{\mu} (c_\mu^\dagger\alpha_\nu + \alpha_\nu c_\mu) ds_\mu \Psi \quad , \tag{158}$$

where δA_ν should otherwise be of the relativistic from

$$\delta A_\nu = \sum_{\lambda\mu} \Gamma^\lambda_{\mu\nu} A_\lambda ds_\mu \quad , \tag{159}$$

if the $\Gamma^\lambda_{\mu\nu}$ are the Ricci coefficients, or

$$c_\mu{}^\dagger\alpha_\nu + \alpha_\nu c_\mu = \sum_{\lambda} \Gamma^\lambda_{\mu\nu}\alpha_\lambda \quad , \tag{160}$$

then

$$c_\mu = g_\mu + i\Phi_\mu \quad , \tag{161}$$

the g_μ commuting with the α_ν and

$$i(\alpha_\nu\Phi_\mu - \Phi_\mu\alpha_\nu) = \sum_{\lambda} \Gamma^\lambda_{\mu\nu}\alpha_\lambda \quad , \tag{162}$$

condition which, in modern Clifford algebra language defines the Φ_μ to be the bivectors associated with the Poincaré group. Think for example of the $\alpha_\nu = \gamma_o\gamma_\mu$ as bivectors in a spacetime "cut" of a vector set γ_ν, then to describe spacetime general connections from point to point the Φ_μ would in general, be $\Phi_\mu = \gamma'_o\gamma'_\mu$ with $\gamma'_\nu = \gamma'_\nu(\underline{x})$ representing a field of spacetime transformations for curved reference spacetime relative to an arbitrary reference γ_ν. Fock and

Iwanenko go on to consider the special case $g_\mu = 0$ and $\Phi = \phi^\mu \alpha_\mu e/\hbar c$ such that ϕ^μ will be considered to correspond to the electromagnetic vector potential and the covariant derivative being written

$$\nabla_\mu \Psi = \left(\frac{\partial}{\partial x_\mu} - \frac{e}{\hbar c} \phi^\mu \right) \Psi \quad , \tag{163}$$

giving a geometric meaning to the electromagnetic field.

V. Fock, in his extense paper "Geometrisierung der Diracschen Theorie des Elektrons"(Zeitschrift für Physik, **55**, 261 - 277 (1929)), clearly states: 1) the geometric meaning of the Dirac γ_μ as vectors, 2) the formulation of the Dirac theory in general relativity, 3) the geometrical meaning of the electromagnetic field as a connection and 4) the gauge theory formulation of gravitation and electromagnetism as geometric fields. Fock had just reviewed the algebra of bispinor quantities and, studying velocity in particular, found relations, which are now known in a general form as Fierz identities, and their transformation properties which he clearly related to the transformation properties of spinors.

The Clifford algebra behind the Fock line of analysis is however not clear and is by far not explicit. The reason for this is that he failed to discover that his "quantum geometry" should in fact be the standard spacetime geometry, or at least a realization of it. It was not until the work of Mercier in the early 1930's that this was clearly seen. An additional problem arised from his analysis being based in the Dirac α_μ matrices and then spacetime vectors were represented by quaternions, which actually correspond to bivectors. The concept of vector and bivector is then mixed throughout the paper. This is common in practice because both the "normalization" $\psi^\dagger \gamma_o \psi$ and the velocity vector $\nu_\mu = \psi^\dagger \alpha_\mu \psi$, are based in considering a reference vector γ_o (or $\alpha_\mu = \gamma_o \gamma_\mu$).

Two basic relations are nevertheless the key point in the analysis of the geometry being constructed; the construction of the local γ-matrices tetrad (here $e_0^2 = -e_1^2 = -e_2^2 = -e_3^2 = 1$)

$$\gamma^\nu_{(x)} = \sum_\beta e_\beta \alpha_\beta h^\nu_\beta(x) \quad , \tag{164}$$

(his equation (23)) and the treatment of those γ^μ matrices as vectors in curved spacetime

$$\nabla_\alpha \gamma^\sigma_{(x)} = \frac{\partial \gamma^\sigma_{(x)}}{\partial \xi^\alpha} + \Gamma^\sigma_{\alpha\rho} \gamma^\rho_{(x)} \quad , \tag{165}$$

(his note to his equation (43)) when he is constructing the Dirac's field energy momentum tensor.

6.5. The Discovery of the Multivector Wave Function

Later in the 30's, came the pioneering papers of Sommerfeld (1939), Sauter (1930), Mercier (1934, 1935), etc. The great difference is a clear distinction of what is geometry and what is a physical proposition. Mercier discussed electrodynamics from the same geometric, Clifford algebra, point of view. The thesis and two papers of Mercier (1934 and 1935) are crucial to formulate the equations of electromagnetism in Clifford algebra. He even went to consider thermodynamics, from this point of view. Mercier had no doubt that he was dealing with spacetime geometry and that the Clifford algebra approach was not to be circumscribed to the relativistic quantum mechanics.

The work of Sauter is important because by recognizing that in the solution to the Dirac equation each of the columns of a $4 \times n$ matrix is a solution of the equation, a minimal left ideal, we could then think of 4×4 matrices as solutions and then the solutions and the operators came to be members of the same algebra of square matrices, now known to represent the spacetime geometric algebra.

Sommerfeld recognizes that the Dirac equation can be solved without any representation of the Dirac algebra and then the solutions are explicitly members of the Dirac algebra. But it was not until the work of Hestenes (1966) that

a) The reference spinor $\phi_H^{(0)}$ was unambiguously defined $i\gamma_0\gamma_1\gamma_2\phi_H^{(0)} = \phi_H^{(0)}$,

b) Real Clifford algebra could be employed and then i was discovered to be an eigenvalue,

c) The geometrical meaning of most quantities could be clearly separated from other types of contributions.

6.6. On the Dirac-Hestenes Equation

We consider here aspects of the multivector form of the electron's wave equation. See (here and additional references in [80]) Proca A. (1930a,b,c), Sauter F. (1930), Mercier A. (1934, 1935), Eddington A.S. (1936), Sommerfeld A. (1939), Riesz M. (1946, 1953, 1958), Quilichini P. (1957), Ravsevskii P.K. (1957), Teitler S. (1965a,b,c, 1966a,b), Hestenes D. (1966, 1975, 1979), Casanova G. (1970, 1976), Boudet R. (1971, 1974, 1985), Salingaros N. and Dresden M. (1979), Greider T.K. (1980), Keller J. (1982a,b, 1984, 1986a,b, 1991), Keller J. and Megy F. (1984), Crawford J.P. (1985), Keller J. and Rodríguez-Romo S. (1991). A recent analysis of some of this work can be found in [6].

6.6.1. The Structure of the Wave Function.

Here we want to show explicitly the multivector content of the Dirac spinor. To start [64] let us consider that associated to each matter field, corresponding to a spin 1/2 particle, there is an energy-momentum field $e_\mu p^\mu(x)$ (summation convention is used). Such that, denoting by $x = e_\mu x^\mu$ points in the observers frame of reference

$$e_\mu p^\mu(x) = m_0 c e_0' \quad , \tag{166}$$

assuming that there exists a (local) frame e_μ' where the energy-momentum is the one corresponding to that of a particle at rest. The frame e_μ' is related to the observers frame e_μ through the local Lorentz transformation

$$e_\mu' = R(x) e_\mu R^{-1}(x), \quad R^{-1} = \tilde{R} \quad , \tag{167}$$

then (166) becomes

$$e_\mu p^\mu(x) = m_0 c R(x) e_0 R^{-1}(x) \quad , \tag{168}$$

we multiply (168) by $R(x)$ on the right

$$e_\mu p^\mu(x) R(x) = m_0 c R(x) e_0 \quad , \tag{169}$$

and use the multivector double projector $P_{+\uparrow}$, with properties

$$P_{+\uparrow} = e_0 P_{+\uparrow} = P_{+\uparrow} e_0 \text{ and } P_{+\uparrow} = P_{+\uparrow} i e_1 e_2 \quad , \tag{170}$$

to obtain

$$e_\mu p^\mu R(x) P_{+\uparrow} = m_0 c R(x) P_{+\uparrow} i e_0 e_1 e_2 \quad . \tag{171}$$

Here the ie_1e_2 factor is to be kept for further reference to the fact that $P_{+\uparrow}$ was chosen as the appropriate projector, other choices could have been made. The up arrow refers to γ_{12} as the direction of spin up and the plus sign to the choice of "positive" mass m_0.

Now assume that there is a function

$$\psi(x) = A(x) R(x) P_{+\uparrow} \ \in \hat{C}_{1,3} \quad , \tag{172}$$

where $\hat{C}_{1,3}$ is the Dirac spacetime algebra and such that [80] can be written, allowing us to use the operator representation $p^\mu = i\hbar\partial^\mu$,

$$\hbar e_\mu \partial^\mu \psi(x) = m_0 c \psi(x) e_0 e_1 e_2 \quad , \tag{173}$$

where the i has been cancelled on both sides of (171). In the reference "rest" frame of the field $R(x) = 1$ and $A(x)$ should be such that $i\hbar e_\mu \partial^\mu A(x) = m_0 c A(x) e_0 e_1 e_2$.

The wave function (172) explicitly contains then 3 main contributions: the existence of the particles' field in $A(x)$, the relative motion of the particles' field in $R(x)$ and the reference to a preferred sign of m_0 and spin in $P_{+\uparrow}$.

This is a derivation from first principles of the, until now semi-phenomenological, Dirac-Hestenes equation [43, 44], this derivation is also an explanation of the geometric reason to consider a multivector equation which goes beyond the multivector analysis which was done, at the beginning of relativistic quantum mechanics, by solving the Dirac equation in terms of multivectors. The $\psi \in \hat{C}_{1,3}$ contains then a, local, Lorentz transformation and the information that a fixed time direction e_0 and a given plane $e_1 e_2$ has been taken as an overall reference. But yet another element of information should be contained in $A(x)$; from the analysis of the bilinear quantity j_0 of the theory, in the normalization consideration $\int_v j_0 d\tau = 1$ where V is some reference volume, $|A(x)|^2$ should have the dimensions of a density, usually called ρ: $A(x)$ then contains a) $a\sqrt{\rho}$ factor, b) a duality rotation (discussed below) and c) a factor f with the effect of the rest mass of the particle which should $f \to 1$ for a massless field.

Quantum mechanics is more general than (173) then our analysis is not a model of all QM, it refers to elementary matter, spin 1/2, fields, as a particular case. The general solution of (173) will give however a Dirac-Hestenes wave function ψ of the standard form mentioned above, that will be discussed in more detail in the following sections, see (187) below, it contains four minimal ideals into one single wave function, but the amount of information is redundant as we will see in section VI.

6.6.2. The Cartan Mapping Applied to the Dirac Equation.

Start now from the Dirac equation for massive spin 1/2 fields

$$i\hbar e_\mu \partial^\mu \Psi(x) = m_0 c \Psi \quad , \tag{174}$$

where $\Psi(x) \in \hat{\mathcal{L}}^{1,3}$ and $e_\mu \partial^\mu \in \hat{C}_{1,3}$ is the gradient operator. Consider again the <u>constant</u> reference spinor $\eta^\dagger$ such that

$$\eta^\dagger \eta = \hat{1} \ ; \ P_{+\uparrow}\eta = \eta \text{ and } \eta^\dagger P_{+\uparrow} = \eta^\dagger \ \text{ then } \ \eta^\dagger = \eta^\dagger P_{+\uparrow} i e_0 e_1 e_2 \quad , \tag{175}$$

and postmultiply (174) by $\eta^\dagger$ on the *l.h.s* . and by $\eta^\dagger P_{+\uparrow} i e_0 e_1 e_2$ on the *r.h.s.* (this is done to introduce into the equations the definitions (175)) to obtain

$$i\hbar e_\mu \partial^\mu \Psi(x) \eta^\dagger = m_0 c \Psi(x) \eta^\dagger i e_0 e_1 e_2 \quad , \tag{176}$$

the obvious definition

$$\psi(x) = \Psi(x)\eta^\dagger \text{ or } \Psi(x) = \psi(x)\eta \quad , \tag{177}$$

brings (176) into (173). The relations (177) correspond to the Cartan map and to the inverse Cartan map respectively ((Crawford 1985) and [105]). For massless fields a different choice of the spinor defined in (175) should be made.

We now use the matrix representation of the geometric superalgebra $\hat{\text{K}}$ [63, 64], an example of $\mathbb{Z}_2$ to illustrate the mathematical structure of (173), its equivalent standard form (174) and the new type of symmetries that are immediate in $\hat{K}$ -algebra.

The basis vectors e_μ which are usually represented by the traceless γ_μ matrices, being elements of order 2 in $\mathbb{Z}_2$, are now represented by the traceless supermatrices

$$e_\mu \to \hat{\gamma}_\mu = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & 0 \end{pmatrix} \quad , \tag{178}$$

The Dirac spinor ψ corresponds in geometric superalgebra to elements of order

$$\Psi \to \hat{\Psi} = \begin{pmatrix} 0 & \Psi \\ 0 & 0 \end{pmatrix} \text{ and } \hat{\Psi}^\dagger = \begin{pmatrix} 0 & 0 \\ \Psi^\dagger & 0 \end{pmatrix} \quad , \tag{179}$$

The reference, spin up, positive mass spinor η obeying $\gamma_0\gamma_{12}\eta = -i\eta$ corresponds to

$$\eta \to \hat{\eta} = \begin{pmatrix} 0 & \eta \\ 0 & 0 \end{pmatrix} \text{ and } \hat{\eta}^\dagger = \begin{pmatrix} 0 & 0 \\ \eta^\dagger & 0 \end{pmatrix} \quad , \tag{180}$$

with $\eta^\dagger = (1\,0\,0\,0)$ in the standard representation of the γ_μ matrices. It has, as mentioned above, the property

$$\hat{\eta}\hat{\eta}^\dagger = \hat{P}_{+\uparrow} \;=\; \frac{1}{2}(\hat{\mathbf{1}} + \hat{\gamma}_0)\frac{1}{2}(\hat{\mathbf{1}} + i\hat{\gamma}_{12}) \quad . \tag{181}$$

Here again (181) is an example of the Cartan map: spinors $\to$ multivectors expressed in $\hat{K}$ algebra.

The standard Dirac equation (free space)

$$i\hbar\gamma_\mu\partial^\mu\Psi = m_0c\Psi \to i\hbar \begin{pmatrix} \gamma_\mu & 0 \\ 0 & 0 \end{pmatrix} \partial^\mu \begin{pmatrix} 0 & \Psi \\ 0 & 0 \end{pmatrix} \;=\; m_0c \begin{pmatrix} 0 & \Psi \\ 0 & 0 \end{pmatrix} , \tag{182}$$

is immediately mapped into the multivector equation, or Dirac–Hestenes equation, thus

a) multiply (182) on the right by $\hat{\eta}^\dagger$ (remind $\hat{\eta}^\dagger$ is constant and $\hat{\eta}^\dagger = \hat{\eta}^\dagger i\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2$)

$$\begin{aligned} i\hbar \begin{pmatrix} \gamma_\mu & 0 \\ 0 & 0 \end{pmatrix} \partial^\mu \begin{pmatrix} 0 & \Psi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \eta^\dagger & 0 \end{pmatrix} = \\ = m_0 c \begin{pmatrix} 0 & \Psi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \eta^\dagger & 0 \end{pmatrix} \begin{pmatrix} i\gamma_{012} & 0 \\ 0 & 0 \end{pmatrix} \quad , \end{aligned} \tag{183}$$

and define $\phi = \Psi\eta^\dagger$ to obtain

$$i\hbar \begin{pmatrix} \gamma_\mu & 0 \\ 0 & 0 \end{pmatrix} \partial^\mu \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} = m_0 c \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} i\gamma_0\gamma_1\gamma_2 & 0 \\ 0 & 0 \end{pmatrix} \quad , \tag{184}$$

an equation where only multivectors occur!, where again the fact that we have chosen for η the reference positive mass and spin up is explicitly shown to obtain the Hestenes equation in superalgebra form:

$$\hbar \begin{pmatrix} \gamma_\mu & 0 \\ 0 & 0 \end{pmatrix} \partial^\mu \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} = m_0 c \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma_{012} & 0 \\ 0 & 0 \end{pmatrix} \quad . \tag{185}$$

It is clear that (185) is block diagonal and the superalgebra here appears as redundant, that is in (185) we don't need, in practice, to use the more general element $\hat{\phi} = \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}$, but the procedures (a) and (b) above are general and other options are open if superalgebra is used and not only the upper left block.

As mentioned above, this mapping would not give back what is considered the more general the standard solution of the Dirac equation in the multivector form. In fact the Hestenes solution for the Dirac-Hestenes equation in free space is often written

$$\psi = \sqrt{\rho} e^{\beta\gamma_5/2} R' \quad , \tag{186}$$

where ρ is a scalar density, β a duality rotation angle and R' a spacetime rotation. As we mentioned this ψ is in fact a multivector with four independent left ideals related in a specific phase, here the bar denotes complex conjugation, which can be represented by a full 4×4 matrix, with the following structure

$$\psi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix} \quad , \tag{187}$$

with

$$\tilde{\psi} = \gamma_0 \psi^+ \gamma_0 = \begin{pmatrix} \bar{\psi}_1 & \bar{\psi}_2 & -\bar{\psi}_3 & -\bar{\psi}_4 \\ -\psi_2 & \psi_1 & -\psi_4 & \psi_3 \\ -\bar{\psi}_3 & -\bar{\psi}_4 & \bar{\psi}_1 & \bar{\psi}_2 \\ -\psi_4 & \psi_3 & -\psi_2 & \psi_1 \end{pmatrix} \quad , \tag{188}$$

and

$$\psi\tilde{\psi} = \Omega_1 + \Omega_2\gamma_5 = \begin{pmatrix} \Omega_1 & 0 & i\Omega_2 & 0 \\ 0 & \Omega_1 & 0 & i\Omega_2 \\ i\Omega_2 & 0 & \Omega_1 & 0 \\ 0 & i\Omega_2 & 0 & \Omega_1 \end{pmatrix} \quad . \tag{189}$$

The Ω_1, and Ω_2 were denoted by δ and $i\pi$ in (141). Then the Cartan mapping should be, as mentioned before,

$$\psi = \sum_{i=1}^{4} \Psi_i \eta_i^\dagger \quad , \tag{190}$$

where each Ψ_i is each one of the columns of (188) and the $\eta_i^\dagger$ are row matrices which have zeros everywhere except at position i. Then a full even multivector is reconstructed in (190). The phases of the Ψ_i are adjusted to give (190) the structure (186).

The direct solution of (185), given by (187) generates $\psi = \sqrt{\rho}e^{\gamma_5\beta/2}R'$ with $R' = Re^{\gamma_1\gamma_2 m_0 c^2 t/h}$ where R is the rotation introduced in (167) above and the exponential in $\gamma_1\gamma_2$ has been the origin of most of the discussions presented below where the fundamental role of the spin bilinear form is considered.

In our theory above we have presented the explanation: spin and action have similar axial properties and proportional to each other, also, for the electron within our geometry the spin density is immediately mapped through the Compton radius $\hbar/2 = |\mathbf{s}| = m_0 c r_{compton}$.

6.7. On the Bilinear Covariants in the Dirac Theory

In the Cartan and inverse Cartan mappings above we have used the well known bilinear covariants, given by (141), in the standard spinor formulation of the Dirac theory, they all correspond to densities of tensor quantities of the general form

$$T_A(x) = \hat{\Psi}\gamma_A\Psi \quad , \tag{191}$$

where the subindex A corresponds to the multivector characters and the γ_A should be Hermitian matrices, that is with a factor $i = \sqrt{-1}$ included whenever necessary. But, in the multivector formulation, there is a second set of bilinear covariants which correspond to quantities which are also relativistic covariants and can be, the same as the $T_A(x)$ above, gauge invariants: they are constructed from the inverse of the wave function multivector

$$\psi^{-1}\psi = \psi\psi^{-1} = 1 \quad , \tag{192}$$

which can be unambiguously defined for the multivector form. In this case we will not be dealing with densities $T_A(x) = \psi\gamma_A\tilde{\psi}$ (corresponding to the T_A of the standard approach as discussed, for example, by Casanova (1976)) but with structural numbers of the theory, of the general form

$$\mathcal{T}_A(x) = \psi\gamma_A\psi^{-1} \quad . \tag{193}$$

They have been discussed by Daviau (1989b) where he found that, from the canonical form of the Dirac-Hestenes wave function, the gauge invariant and covariant structural numbers of the theory are

$$\begin{aligned} f_0 \;&=\; \psi\gamma_0\psi^{-1} = \sqrt{\rho}e^{\frac{-\beta}{2}\gamma_5}R\gamma_0\frac{1}{\sqrt{\rho}}e^{\frac{\beta}{2}\gamma_5}\tilde{R} \\ &=\; e^{\frac{\beta}{2}\gamma_5}R\gamma_0\tilde{R}e^{\frac{-\beta}{2}\gamma_5} = e^{\beta\gamma_5}e_0 = e_0e^{-\beta\gamma_5} \quad , \end{aligned} \tag{194}$$

$$\frac{2}{\hbar}S = \psi\gamma_1\gamma_2\psi^{-1} = \frac{1}{k_1}F = e_1e_2 \quad , \tag{195}$$

$$f_3 = \psi\gamma_3\psi^{-1} = e^{\beta\gamma_5}e_3 = e_3e^{-\beta\gamma_5} \quad . \tag{196}$$

This $\mathcal{T}_A$ covariants should really be taken as intensive quantities, they do not depend on the electron density ρ. In terms of Daviau these quantities are remarkable because their intensity is rigorously the same both where the electron field has a high density or where it has a vanishing density. They can be studied in the multivector formalism because it is in this formalism where ψ is invertible.

A quantity like $\mathcal{T}_0 = f_0$ is the one that seems, in principle, associated to the mass term of the electron theory. In the analysis below a mapping of the Dirac equation will be presented where this is discussed and constrasted with the approach of Daviau and Lochak (1991) (see also Daviau 1993) where they use densities of bilinear forms and then have to speak in terms of “a variable mass term”.

As in the case of the ordinary Dirac bilinear forms, so well known because of the Pauli-Fierz identities, we can use the inverse Cartan map on the intensive bilinears and start from the Dirac spinor and the corresponding mapping of the inverse Dirac multivector:

$$\psi^{-1}\eta_i = \Psi_{(\psi^{-1})} \text{ and then } \mathcal{T}_A = \bar{\Psi}_{(\psi^{-1})}\gamma_A\Psi \quad . \tag{197}$$

This mapping will be discussed elsewhere.

6.8. The Bivector-Spinor Mapping of Campolattaro and Daviau

There were two series of papers, which were almost equivalent, developing the idea of a mapping of the Dirac equation into a Maxwell like form, it is that the Dirac equation could be written in the form $\Box F = 0$. And, simultaneously, introduced the idea that the basic Maxwell equation

$$\Box F = \frac{4\pi}{c} J \quad , \tag{198}$$

which in multivector algebra is a relation between the divergence of the bivector F and the vector J can be mapped into a Dirac like equation for an "spinor" ψ which can generate the bivector F in the **multivector** bilinear map

$$\psi\gamma_1\gamma_2\psi^* = F \quad . \tag{199}$$

The equation takes a special simple form in the case of $J = 0$.

The work of Campolattaro (1980) started with the analysis of the Maxwell equation by writing F in the equivalent bilinear form $\bar{\Psi}\gamma_\mu\gamma_\nu\Psi$ corresponding to (199) above, and by simple replacement into the Maxwell equation (198) together with the use of the constant $\bar{\Psi}\Psi = b$, he obtains the Dirac like, non linear, equation for Ψ.

In his two following papers, Campolattaro (1990), pursuits his mapping and shows that the inverse procedure can be used for the Dirac equation, to arrive at the Maxwell like equation obeyed by an $\mathcal{F}$ defined in exactly the same form as equation (198).

In what can be thought as a completely independent work Daviau proposed in 1988 that both the Dirac and the Maxwell equations could be generalized and mapped into each other in such a way that it is clear that they are mathematically equivalent (provided we fix the solutions by the use of subsidiary constrains) and interchangeable for the Dirac or for the Maxwell fields, Daviau (1989). This paper makes full use of spacetime Clifford algebra whereas the work of Campolattaro considers the Dirac matrix ring algebra and spinors, F is represented by him in component tensor form.

6.8.1. The Campolattaro Reinterpretation of the Dirac Equation for the Free Electron.

Let us consider the Dirac equation for the free electron, i.e.,

$$(\gamma^\mu \partial_\mu + im)\Psi = 0 \quad . \tag{200}$$

By multiplying equation (109) on the left by $\bar{\Psi}\gamma^\nu$ one has ($\gamma^{\nu\mu} = -\gamma^{\mu\nu}$)

$$\bar{\Psi}\gamma^\nu \gamma^\mu \partial_\mu \Psi + im\bar{\Psi}\gamma^\nu \Psi = 0 \quad . \tag{201}$$

Using $S^{\mu\nu} = \frac{i}{2}\gamma^{\mu\nu}$ and $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} + \gamma^{\mu\nu}$, equation (201) reads

$$2i\bar{\Psi}S^{\mu\nu}\Psi_{,\mu} + \eta^{\mu\nu}\bar{\Psi}\partial_\mu\Psi + im\bar{\Psi}\gamma^\nu\Psi = 0 \quad . \tag{202}$$

By taking the Hermitian conjugate of equation (111), one has

$$2i\bar{\Psi}_{,\mu}S^{\mu\nu}\Psi - \eta^{\mu\nu}(\partial_\mu\bar{\Psi})\Psi + im\bar{\Psi}\gamma^\nu\Psi = 0 \quad , \tag{203}$$

and by adding equations (111) and (112), one obtains, for the antisymmetry of $S^{\mu\nu}$,

$$(\bar{\Psi}S^{\nu\mu}\Psi)_{,\mu} = \eta^{\mu\nu} Im(\bar{\Psi}\Psi_{,\mu}) + m\bar{\Psi}\gamma^\nu\Psi \quad . \tag{204}$$

Similarly, by multiplying equation (109) on the left by $\bar{\Psi}\gamma^5\gamma^\nu$ and by repeating the steps followed in the previous lines, one has

$$(\bar{\Psi}\gamma^5 S^{\nu\mu}\Psi)_{,\mu} = \eta^{\mu\nu} Im(\bar{\Psi}\gamma^5\Psi_{,\mu}) \quad . \tag{205}$$

Equations (204) and (205) are completely equivalent to the Dirac equation (200). Therefore, one has, by using the results expressed by equations (200-205) above, that the Dirac equation is equivalent to the Maxwell equations for an "electromagnetic field" $\bar{F}^{\mu\nu}$ defined by

$$\bar{F}^{\mu\nu} = \bar{\Psi}S^{\mu\nu}\Psi \quad , \tag{206}$$

and thence

$$^*\bar{F}^{\mu\nu} = \bar{\Psi}\gamma^5 S^{\mu\nu}\Psi \quad , \tag{207}$$

generated by the two currents

$$j^\mu = \eta^{\mu\nu} Im(\bar{\Psi}\Psi_{,\nu}) + m(\bar{\Psi}\gamma^\mu\Psi) \quad , \tag{208}$$

and

$$g^\mu = \eta^{\mu\nu} Im(\bar{\Psi}\gamma^5\Psi_{,\nu}) \quad , \tag{209}$$

the first electronic in nature and the second magnetic monopolar, or simply monopolar. The gauge conditions are automatically satisfied because each of the four components of the Dirac spinor satisfies the Klein-Gordon equation and the current $m\bar{\Psi}\gamma^\mu\Psi$ is conserved.

6.8.2. The Work of Daviau and the Analysis of Rodrigues et al.

As the use of spacetime Clifford algebra (SCA) allows a more comprehensive treatment, or at least a simple writing, we will follow the Daviau approach. He starts by showing that a matrix representation of the spacetime Clifford algebra allows a straightforward analysis of several relations in (SCA). He analyses scalars, vectors, bivectors, axial vectors and pseudoscalars in some detail in order to have a quick reference to the relationship among them. He also writes the gradient operator for spacetime, in matrix form, and describes the dot product of that operator with each of the multivector fields. He can then write the analysis (which was already known some 60 years before from the work of Mercier (1934)), to proceed to the, for him, obvious spacetime generalization of the Maxwell equation for the case where:

a) the current is enlarged to have both a vector and trivector parts $J = J_v + J_t$. As we know the trivector part, dual of the vector part, can be interpreted in Maxwell theory as a current of magnetic monopoles in the sense that the electric and magnetic fields are dual to each other.

b)In a second step Daviau generalizes the concept of the bivector F to study the possible inclusion of the full even part of the spacetime algebra $F = F_s + F_b + F_p$, which, in the analysis of the equivalent to spinors in spacetime algebra, corresponds to a spinor which can be written $\psi = \sqrt{\rho} e^{\gamma_5 \beta/2} R$ and interpreted as a weight $\sqrt{\rho}$, a duality rotation phase $e^{\gamma_5 \beta/2}$ and a Lorentz transformation R. Here no mass term $e^{\gamma_1 \gamma_2 m_0 c^2 t/\hbar}$ is needed.

The two successive generalizations yield both a generalized Maxwell theory and the possibility of interpreting F as generated by the Ψ and then the possible mapping of the Maxwell equation into a Dirac like equation for Ψ.

But before doing this mapping he first proceeds to map the Hestenes form of the Dirac equation back into a Maxwell like form by interpreting the solution of the Dirac equation as the sum of a scalar, an electric, a magnetic and a pseudoscalar part.

The algebraic manipulations behind this mappings are much better understood following the analysis presented by Rodrigues, Vaz and Recami (1993), Vaz and Rodrigues (1993) and Rodrigues and Vaz (1994) because they start not by mapping the equations but by examining the consequences of the free Maxwell equations $\Box F = 0$ when $F = b\psi\gamma^1\gamma^2\psi^*$.

We will follow the analysis of Rodrigues and Vaz (RV) starting from the multivector

$$F = H e^{\beta\gamma_5} R \gamma_1 \gamma_2 R^* \quad , \tag{210}$$

and for the electromagnetic field writing $H \equiv b\rho$ with $\rho \leq 0$, we then have the formal

$$F = b\psi\gamma_1\gamma_2\psi^* \quad , \tag{211}$$

the bivector which we consider to obey the free Maxwell equation:

$$\Box F = 0 \quad , \tag{212}$$

RV then look for solutions of eq. (212) of the form (120); they consider that since eq. (211 is valid when F is non-null ($F^2 \neq 0$) then plane-wave solutions of eq.(212) are excluded (since in this case $F^2 = 0$). This is already an important consideration about the difference between the Dirac and Maxwell fields even if both can be described by (212) for the source free case. RV proceed to consider the trivial non-null solution of eq.(212) given by $F =$ constant, the case when b and ρ in eq.(211) are constants or related in a special form. They suppose first that b, ρ and β be constants. From eq.(211) in eq.(212), RV obtain the non-linear *Heisenberg-like* spinor equation:

$$\Box\psi\gamma_1\gamma_2 + \mathcal{F}(\psi) = 0, \{b, \rho\} = cte. \quad , \tag{213}$$

with

$$\mathcal{F}(\psi) = \gamma^\mu\psi\gamma_1\gamma_2(\partial_\mu\psi^*)\psi(\psi\psi^*)^{-1} \quad . \tag{214}$$

From $RR^* = R^*R = 1$ it follows that $(\partial_\mu R)R^* + R(\partial_\mu R^*) = 0$, and Hestenes [47] writes

$$\partial_\mu R = \frac{1}{2}\Omega_\mu R \quad , \tag{215}$$

where ($\Omega_\mu + \Omega_u^* = 0$ or $\Omega^*{}_\mu = -\Omega_\mu$)

$$\Omega_\mu = 2(\partial_\mu R)R^* \quad . \tag{216}$$

Since we have supposed ρ and β constant, eq.(124) can be written as

$$\partial_\mu\psi = \frac{1}{2}\Omega_\mu\psi \quad . \tag{217}$$

using $\partial_\mu R^* = R^*(R\partial_\mu R^*)\frac{2}{2}$.
If we introduce eq.(217) into eq.(214) and define the intensive, structural, multivector bilinear, 2-form S as

$$S \equiv \frac{\hbar}{2}R\gamma_1\gamma_2R^* = \frac{\hbar}{2}\psi\gamma^1\gamma^2\psi^{-1} \quad , \tag{218}$$

where the constant $\hbar$ is identified by RV with the (reduced) Planck constant, then eq.(213) acquires the relevant (non-linear) form:
$-\frac{1}{\hbar}S\Omega_\mu = \frac{\hbar}{2}R\gamma_1\gamma_2R^*R(\partial_\mu R^*)2\frac{1}{\hbar}$, an intensive or structural relation which, because $\{b,\rho\} = cte$, can be written

$$\Box\psi\gamma_1\gamma_2 - \frac{1}{\hbar}\gamma^\mu S\Omega_\mu\psi = 0 \quad . \tag{219}$$

RV point out that eq.(128) is an interesting result: it is equivalent to the free Maxwell equations (121), under the above assumptions. The Daviau covariant S corresponds to spin.

Given that both S and Ω_μ are 2-forms, the product $S\Omega_\mu$ in eq.(128) results in the sum of a scalar, a 2-form and a pseudo-scalar; that is:

$$S\Omega_\mu = -\rho_\mu + E_{\mu,\alpha\beta}(\gamma^\alpha\wedge\gamma^\beta) + \gamma^5 r_\mu \quad , \tag{220}$$

where ρ_μ, $E_{\mu,\alpha\beta}$ and r_μ are scalars. Consider as a guide the counterpart of an electron at rest in the Dirac-Hestenes equation, then $R = e^{-\gamma_1\gamma_2 m_o c^2 t/\hbar}$ and ($R^* = \tilde{R} = e^{+\gamma_1\gamma_2 m_o c^2 t/\hbar}S = \frac{\hbar}{2}R\gamma_1\gamma_2\tilde{R} = \frac{\hbar}{2}\gamma_{12}$) also $\Omega_o = 2(\partial_o R)R^* = 2m_o c\gamma_1\gamma_2/\hbar; \Omega_i = 0, i = 1,2,3$ and $S\Omega_o = m_o c, S\Omega_i = 0$ or $\hbar\partial\psi\gamma_{12} - m_o c\psi\gamma^o = 0$.

Then the product $S\Omega_\mu$ reduces to only one scalar constant and (219) becomes the Dirac-Hestenes equation, the same is true if a more general R is used than that of an electron at rest. The more general analysis of RV follows:

Let us first suppose that $S\Omega_\mu$ possesses only a scalar part; then

$$\gamma^\mu S\Omega_\mu = -\rho_\mu\gamma^\mu \equiv -\rho \quad . \tag{221}$$

Now, given the velocity field v, defined as

$$v \equiv R\gamma^0 R^* \quad , \tag{222}$$

so that $\rho v = \psi\gamma^0\psi^*$, let us define the mass m in such a way that

$$\rho\psi \equiv e^{\beta\gamma^5} mcv\psi \quad , \tag{223}$$

from where it follows:

$$\rho\psi = mc\psi\gamma^0 \quad . \tag{224}$$

When we insert eq.(221) into eq.(219) and then use eq.(224), we eventually end up with a *linear* equation:

$$\Box\psi\gamma_1\gamma_2 + \frac{mc}{\hbar}\psi\gamma^0 = 0 \quad , \tag{225}$$

which is just the Dirac-Hestenes equation for a free particle (electron).

At this point RV go beyond the 1980 Campolattaro analysis and introduce what is in fact a mathematical basis to consider interaction fields. In fact, the equations below could also be derived by gauging. Campolattaro considered this extension in Campolattaro (1990a and b). Now RV go on assuming b, ρ and β to be still constant but, instead of supposing that $S\Omega_\mu$ possesses only a scalar part, they use its general expression, eq.(220):

$$E_{\mu,\alpha\beta}\gamma^\mu(\gamma^\lambda \wedge \gamma^\beta) = -\frac{e}{c}A_\mu\gamma^\mu - \gamma^5\frac{g}{c}B_\mu\gamma^\mu = -\frac{e}{c}A - \gamma_5\frac{g}{c}B \quad , \tag{226}$$

where A_μ and B_μ are

$$\frac{e}{c}A_\mu \equiv \eta_{\nu\sigma}E_{\nu,\mu\sigma} \quad , \tag{227}$$

$$\frac{g}{c}A_\mu \equiv \eta_{\mu\nu}\epsilon_{\nu\sigma\rho\tau}E_{\sigma,\rho\tau} \quad , \tag{228}$$

with $\epsilon_{\nu\sigma\rho\tau} = +1(-1)$ for even (odd) permutations of [0,1,2,3] while $\epsilon_{\nu\sigma\rho\tau} = 0$ when two indexes are equal. Here A and $\gamma^5 B$ play the role of electromagnetic potentials when *external* (electromagnetic) fields are present: the potentials associated to an electric charge e and a magnetic monopole $\gamma^5 g$, respectively. Here we can remind, as we have mentioned above (Sutherland (1989)) that there is a gauge for the Dirac equation where the mass term disappears explicitly. They now define, in analogy with eq. (223),

$$r\psi \equiv e^{\beta\gamma^5}\mu c v\psi = \mu c\psi\gamma^0 \quad , \tag{229}$$

where $r = r_\mu\gamma^\mu$, then eq.(219) assumes the form:

$$\Box\psi\gamma_1\gamma_2 + (m + \gamma_5\mu)\frac{c}{\hbar}\psi\gamma^0 + (eA + \gamma_5 gB)\frac{1}{\hbar c}\psi = 0 \quad . \tag{230}$$

A later analysis shows that ρ and β are constants only if A and B are also constants in (226).

6.8.3. The Equivalence of the Campolattaro and Daviau.

Since the early work of Fock and Iwanenko it is clear that spinor formulations and multivector spinor formulations can be mapped into each other, although they are not strictly equivalent as we have discussed above in the previous chapters. Vaz and Rodrigues (1993) have recently shown the faithfulness of the mapping, we present here the main steps they followed. Vaz and Rodrigues

(1993) wrote in detail the proof of the equivalence between the non linear (multivector) spinor equation for the Maxwell field and the equivalent equation derived by Campolattaro for the Dirac equation to transform it into a Maxwell-like equation, which are in fact different because Campolattaro uses Ψ which is the Dirac column spinor and Vaz and Rodrigues (Daviau) use a Hestenes matrix spinor with four columns. The analysis of Vaz and Rodrigues goes as follows:

a. They write the Dirac spinor in the column matrix representation

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \quad , \tag{231}$$

$$\bar{\Psi} = (\bar{\psi}_1, \bar{\psi}_2, -\bar{\psi}_3, -\bar{\psi}_4,) \quad . \tag{232}$$

b. They substitute into the Campolattaro equation

$$\gamma^\mu \partial_\mu \Psi = -i\gamma^\mu \frac{e^{\gamma^5 \alpha}}{\rho} \{Im(\partial_\mu \bar{\Psi}\Psi) - \gamma_5 Im(\partial_\mu \bar{\Psi}\gamma_5 \Psi)\}\Psi \quad , \tag{233}$$

and then they transform that equation into an equation for column matrices

$$\begin{aligned} Im(\partial\bar{\Psi}\Psi) &= \tfrac{-i}{2}[(\partial\bar{\psi}_1\psi_1 - \bar{\psi}_1\partial\psi_1) + (\partial\bar{\psi}_2\psi_2 - \bar{\psi}_2\partial\psi_2) \\ -(\partial\bar{\psi}_3\psi_3 - \bar{\psi}_3\partial\bar{\psi}_3) - (\partial\bar{\psi}_4\psi_4 - \bar{\psi}_4\partial\bar{\psi}_4)] &\equiv \tfrac{-i}{2}\xi \quad , \end{aligned} \tag{234}$$

$$\begin{aligned} Im(\partial\bar{\Psi}\gamma_5\Psi) &= \tfrac{-1}{2}[(\partial\bar{\psi}_1\psi_3 + \bar{\psi}_3\partial\psi_1) + (\partial\bar{\psi}_2\psi_4 + \bar{\psi}_4\partial\psi_2) \\ -(\partial\bar{\psi}_3\psi_1 - \bar{\psi}_1\partial\psi_3) - (\partial\bar{\psi}_4\psi_2 + \bar{\psi}_2\partial\psi_4)] &\equiv \tfrac{-1}{2}\eta \quad , \end{aligned} \tag{235}$$

to obtain

$$\{Im(\partial_\mu \bar{\Psi}\Psi) - \gamma_5 Im(\partial_\mu \bar{\Psi}\gamma_5\Psi)\}\Psi = \frac{-i}{2} \quad \xi \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} + \eta \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \quad . \tag{236}$$

Thus equation (233) is explicitly

$$\gamma^\mu \partial_\mu \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = -\frac{1}{2}\gamma^\mu \frac{e^{\gamma^5 \alpha}}{\rho} \begin{pmatrix} \xi\psi_1 + \eta\psi_3 \\ \xi\psi_2 + \eta\psi_4 \\ \xi\psi_3 + \eta\psi_1 \\ \xi\psi_4 + \eta\psi_2 \end{pmatrix} = -\frac{1}{2}\gamma^\mu \frac{e^{\gamma^5 \alpha}}{\rho} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{pmatrix} \quad . \tag{237}$$

c. For the non linear equation of Vaz and Rodrigues

$$\gamma^\mu \partial_\mu \psi \gamma_1 \gamma_2 + \mathcal{F}(\psi) = 0 \quad , \tag{238}$$

with

$$\mathcal{F}(\psi) = \gamma^\mu \psi \gamma_1 \gamma_2 (\partial_\mu \psi^*) \psi (\psi^* \psi)^{-1} \quad , \tag{239}$$

they write the ψ in full

$$\psi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix},$$

$$(240)$$

$$\psi^* = \begin{pmatrix} \bar{\psi}_1 & \bar{\psi}_2 & -\bar{\psi}_3 & -\bar{\psi}_4 \\ -\psi_2 & \psi_1 & -\ \psi_4 & \psi_3 \\ -\bar{\psi}_3 & -\bar{\psi}_4 & \bar{\psi}_1 & \bar{\psi}_2 \\ -\ \psi_4 & \psi_3 & -\psi_2 & \psi_1 \end{pmatrix} \quad ,$$

and by direct substitution they obtain a 4×4 matrix equation

$$\gamma^\mu \partial_\mu \psi = -\frac{1}{2} \gamma^\mu \frac{e^{-\gamma^5 \beta}}{\rho} \chi \quad , \tag{241}$$

or, in terms of components,

$$\gamma^\mu \partial_\mu \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix} =$$

$$(242)$$

$$= -\frac{1}{2} \gamma^\mu \frac{e^{-\gamma^5 \beta}}{\rho} \begin{pmatrix} \chi_1 & -\bar{\chi}_2 & \chi_3 & \bar{\chi}_4 \\ \chi_2 & \bar{\chi}_1 & \chi_4 & -\bar{\chi}_3 \\ \chi_3 & \bar{\chi}_4 & \chi_1 & -\bar{\chi}_2 \\ \chi_4 & -\bar{\chi}_3 & \chi_2 & \bar{\chi}_1 \end{pmatrix} \quad .$$

Comparison of (237) with (6.8.3) shows that every column of (6.8.3) obeys the Campolattaro equation. But it should be stressed that the ψ of Daviau is a multivector and as such it has additional geometrical content.

The mapping of the Daviau-Vaz-Rodrigues F into the Campolattaro $F^{12} = \bar{\Psi}\gamma^{12}\Psi$ is not reduced to what was presented in (232-6.8.3) above, because $\psi = \sum_i \Psi_i \eta_i^\dagger$ as discussed in IV and then

$$F = (\sum_j \eta_j \bar{\Psi}_j)\gamma^{12}(\sum_i \Psi_i \eta_i^\dagger) \quad , \tag{243}$$

and then, besides the $\bar{\Psi}_i\gamma^{12}\Psi_i$ terms other 12 terms $\bar{\Psi}_j\gamma^{12}\Psi_i, i \neq j$ are to be considered for the construction of the multivector F. We repeat, the multivector F contains more information than the minimum needed to satisfy the Campolattaro equations, equivalent to the Dirac equation by construction. As presented above the F is equivalent to the multivector Dirac-Hestenes equation, related but not linearly dependent to the Dirac equation. The discussion in Part II of this paper will analyze the advantages for a geometrical analysis of the Dirac-Hestenes spinor.

6.9. Non-Dispersive de Broglie Wave-packets from the Maxwell Equations

In an interesting extension of their analysis Rodrigues, Vaz and Recami (1993) continue the study of $\Box F = 0$ when the canonical form $\psi = be^{\gamma^5\beta/2}R'$ is not restricted.

RVR pass to the most general case, by *eliminating* the restriction that b is constant. To obtain:

$$\Box\psi\gamma_1\gamma_2 + \mathcal{F}(\psi) = -\Box(\log b)\psi\gamma_1\gamma_2 \quad , \tag{244}$$

which *generalizes* the non-linear equation (213), letting ρ and β to be variables here, they get

$$\partial_\mu\psi = [\partial_\mu \log(\rho e^{\beta\gamma_5})^{1/2}]\psi + \frac{1}{2}\,\Omega_\mu\psi \quad , \tag{245}$$

which, using eq.(218), results in:

$$\Box\psi\gamma_1\gamma_2 - \frac{1}{\hbar}\gamma^\mu S\Omega_\mu\psi = -(\Box \log b)\psi\gamma_1\gamma_2 - [\Box \log(\rho e^{\beta\gamma_5})^{1/2}]\psi\gamma_1\gamma_2 \quad . \tag{246}$$

RVR rewrite the 1.h.s. of eq.(246) in order to have

$$[\Box R\gamma_1\gamma_2 - \tfrac{1}{\hbar}\gamma^\mu S\Omega_\mu R](\rho e^{\beta\gamma_5})^{1/2} = -(\Box \log b)\psi\gamma_1\gamma_2 - 2[\Box \log(\rho e^{\beta\gamma_5})^{1/2}]\psi\gamma_1\gamma_2 \quad . \tag{247}$$

The 1.h.s. of eq.(247) vanishes, once R is required to satisfy eq. (219) (which was written in terms of ψ because ρ, β were there supposed to be constants); then we must have [K being a constant]:

$$\Box \log b = -2\Box \log(\rho e^{\beta\gamma_5})^{1/2} \Longleftrightarrow b = \frac{K}{\rho e^{\beta\gamma_5}} \quad , \tag{248}$$

which implies that F is proportional to $R\gamma_1\gamma_2 R^*$. Equation (248), therefore, implies a (non-null) constant field F . Notice, incidentally, that in eq.(248) it must be either $\beta = 0$ or $\beta = \pi$, since b is a *scalar*; and this is a consequence of supposing R to obey the Dirac-Hestenes equation.

Putting eq.(248) into eq.(246), RVR get finally the generalized spinor equation:

$$\Box\psi\gamma_1\gamma_2 - \frac{1}{\hbar}\gamma^\mu S\Omega_\mu\psi = (\Box \log\psi_0)\psi\gamma_1\gamma_2 \quad , \tag{249}$$

with $\psi_0 \equiv (\rho e^{\beta\gamma_5})^{1/2}$. And again, if $S\Omega_\mu$ has only a scalar part, eq.(249) can be written -according to our previous discussion- as:

$$\Box\psi\gamma_1\gamma_2 + \frac{mc}{\hbar}\psi\gamma^0 = (\Box \log\psi_0)\psi\gamma_1\gamma_2 \quad , \tag{250}$$

which is a (non-linear) generalized Dirac-Hestenes equation. RVR show that, if one applies the Dirac operator $\Box$ to the above equation, one obtains [$\Box^2 \equiv \partial^2$]:

$$\Box^2\psi + \left(\frac{mc}{\hbar}\right)^2\psi = \frac{\Box^2\psi_0}{\psi_0}\psi + W\psi \quad , \tag{251}$$

where

$$W\psi = \eta^{\mu\nu}(\partial_\nu \log\psi_0)\Omega_\mu\psi \quad . \tag{252}$$

The term $W\psi$ can be easily calculated in the rest frame; the result is that it vanishes, i.e.:

$$W\psi = 0 \quad . \tag{253}$$

Then eq.(251) assumes the interesting, simple form:

$$\Box^2\psi + \left(\frac{mc}{\hbar}\right)^2\psi = \frac{\Box^2\psi_0}{\psi_0}\psi \quad . \tag{254}$$

This is a *non-linear* Klein-Gordon equation, which exactly coincides with the equation proposed by Gueret and Vigier (1982), and possesses *localized*, *non-dispersive* solutions. The term $\Box^2\phi_0/\phi_0$ is usually called the "quantum potential". Following Gueret and Vigier (1982) and also Mackinnon (1981) it can be proposed

$$\Box^2\phi_0 = \left(\frac{mc}{\hbar}\right)^2 \phi_0 \quad , \tag{255}$$

then:

$$\Box^2\phi = 0 \quad , \tag{256}$$

which is just the case discussed by Mackinnon (1981). In particular, eq.(256) admits a non-trivial solution, representing a non-dispersive *soliton* (localized wave-packet) which moves *undeformed* with subluminal speed.

Let us recall, here, that already in 1915 Bateman had looked for "solitonic" solutions of Maxwell equations.

RVR further notice that -when we replace $S\Omega_\mu$ in eq.(249) by its full expression, eq.(220), containing a scalar, a 2-form and a pseudo-scalar part- then eq.(249) gets its most general form:

$$\Box\psi\gamma_1\gamma_2 + (m+\gamma_5\mu)\frac{c}{\hbar}\psi\gamma^0 + (eA+\gamma_5 gB)\frac{1}{\hbar c}\psi = (\Box\log\psi_0)\psi\gamma_1\gamma_2 \quad . \tag{257}$$

Actually, eq.(139) is a particular case of eq.(153), valid when the non-linear term $(\partial\log\psi_0)\psi\gamma_1\gamma_2$ can be neglected.

6.10. The Constrains of the Maxwell-like Bivectors

Given an electromagnetic field F, from the Rainich (Misner-Wheeler) theorem is deduced that

$$F = b\psi\gamma_1\gamma_2\psi^* \quad , \tag{258}$$

where ψ is a multivector spinor field (whose canonical form is $\psi = \rho^{1/2}e^{\gamma_5\beta/2}R$). Rodrigues and Vaz showed that even supposing b, ρ and β to be *not* constant, correctly represents the electromagnetic field, the field solves the *free* Maxwell equations (without sources) $\Box F = 0$.

On the other hand, F can be written:

$$F = K\Phi\gamma_1\gamma_2\Phi^* \quad , \tag{259}$$

where $b \equiv K/\rho e^{\beta\gamma_5}$; and where $\Phi \equiv RA_o(t)$ is a Dirac-Hestenes spinor field which satisfies the (linear) Dirac-Hestenes equation [as it is discussed by Hestenes (1966) where he uses the fact that any $R \in Spin_+(1{,}3)$ can be written as the exponential of a bivector].

Those spinor fields are then related by the rest mass factor

$$A_o(t) = e^{\gamma^{12} m_o c^2 t/\hbar} \quad , \tag{260}$$

and in particular, for the "electron solutions" ($i.e.,$ for$\beta = 0$), by:

$$\psi = \rho^{1/2}\Phi \quad , \tag{261}$$

which coincides, as shown by RV, with a well-known expression in de Broglie's theory of double solution. [For the positon, one would get $\psi = -\gamma^5\rho^{1/2}\Phi$]. Let us recall that in such de Broglie's theory ψ was the electron (total) quantum-probabilistic wave function, but Φ (which obeys the Dirac-Hestenes equation) was a physical wave!

An important remark of the RV approach, is that the DH spinor field Φ is just the *rotor* part of ψ and has therefore only 6 degrees of freedom, so as the electromagnetic field F (while ψ possesses 8 degrees of freedom: and, in the case, the spinor field Φ could be ultimately of electromagnetic nature, as suggested by equations (258) and (261). For instance, the basic equation (211), or (258), shows the strict relation existing between the non-null electromagnetic field F (present in the absence of sources!) and the electron wave-function ψ.

They conclude that many of de Broglie's ideas concerning the interpretation of quantum mechanics should be seriously revisited, while the language of Clifford algebras appears to be particularly convenient for that purpose.

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