

COMMENT

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**ALGEBRAS: OLD AND NEW AND THEIR USE.**

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We know that it was during the ruling of Califa Al-Ma'mūn (813-833) that the wise man Mahammad ben Mūsa al-Khwārizmi, a member of the Dā al-Hikma or "house of wisdom" (a sort of scientific academy of the epoch) in Baghdad, wrote two main books, the al-jabr which treats the art of adding equal terms on both sides of an al-karaji (now known as equations) and a book on al-mucabala about the art of subtracting equal amounts to both sides of an al-karaji in order to eliminate negative terms. This is, as far as we know, the state of the art of that we now know as algebra, in the times of the kingdom of Baghdad in the scientific school founded by orders of the legendareous Harūm al-Rashid.

Extremely important was that in the very origins of these sciences stood geometrical concepts. When in chapter one of al-jabr the famous al-Khwārizmi wrote equations like  $x^2 = 40x - 4x^2$ , which he transformed into  $x = 8$  he really meant, what is extremely clear in his writings because words were used not symbols, that he was analyzing a square surface of side 8 units comparing it to a surface  $8 \times 40$  square units. This becomes even more clear in the book when in the following chapters the art of measuring lengths areas and volumes, including circular areas or cylindrical volumes, is discussed.

Then algebra was related from its very creation, to a multilineal problem, where lengths, areas and volumes corresponded to simultaneous but independent variables. Most probably algebra was in fact born from the need to solve definite practical, geometric and accounting, problems. The abstraction came afterwards and the use of algebraic relations to solve other problems, as

a mathematical model is still in progress. The new problems on turn have forced further developments of algebra.

Geometrical relations were combined in an algebra but not created by the arabs from Baghdad, its suffices to remember that the Greeks or the Hindus knew, long before al-Khwārizmi, quite reasonable approximations for numbers like  $\pi$  (Archimedes or Heron used  $\pi = 31/7$  and Brahmagupta (7th century A.D.) used  $\pi = \sqrt{10}$ ) although the arabs had an incredible good value  $\pi = 62832/20000$  and the chinese (Lin Hui, 3rd century) knew also the equivalent to our approximate value  $\pi = 3.1416$ .

In fact we know from the oldest writings that an approximation to the integer numbers algebra was known by the ancient people and that the Hindus and Persians could have invented the zero in order to solve problems such as: if I have three goats and I want to have n how many more goats do I need? and even the negative integers, thus extending the “natural” numbers, in answering questions like: if I have five goats and I want to have three, how many goats do I need? In a clear understanding that the answer to: how many more do I need? is basically different from the answer to a question like: if I want to have less, how many do I have to subtract?

Even an algebra for rational numbers or for a primitive concept of real numbers was somehow handled by the old civilizations.

But it was certainly when the Arabs wanted to study the partitioning of length, areas and volumes, with a careful handling of a length unit as a basic unit, that algebras, in a way similar to our present understanding, were born.

Writings of other authors like Tabit ben Qurra (836-901) were clear that quadratic equations related to geometry. Or in the XI century Omar Khayyam in his 1074 book “On the Proofs of the Problems of Algebra and Mucabala” spoke about cubic equations as relations of volumes and the classification he made of those equations was in essence a form of knowing how two relate a “cube” to a partitioning of it into a length times a square of another length and an area times other length.

When in the XIII century the Italians masters translated al-jabr into latin, these “maestri d’abbaco”, for example Gerardo di Cremona or Leonardo di Pisa (called the Fibonacci), considered that algebra was related to geometrical

concepts. In fact after Fibonacci's fundamental Liber Abacci (1202) or "the art of computing" the book Practica Geometriae (1220) or Liber Quadratorum (1225) appeared. In the second edition of Liber Abacci (1228) equations like  $xy + z = 30$  or  $3x + 2y + \frac{1}{2}z = 30$  really meant the equality of volumes as was explicitly stated in the names given to the variables in the prosa which described the equations.

Two centuries after Fibonacci the "maestri Fiorentini" (XV century) like Benedetto, Biaggio or Antonio Mazzinghi clearly kept the relation between algebra and geometry. A century afterwards in the school mastered by Scipione del Ferro (1526), Niccolo Tartaglia, Gerolamo Cardano (1539) and their students gave a step forward when they considered new types of abstract mathematical problems:  $a + b = 10$  and  $ab = 40$  required answers like  $a = 5 + \sqrt{-15}$  and  $b = 5 - \sqrt{-15}$  which in fact was the origin of a new concept, called "meno" by them. In fact "meno" corresponded to the square root of minus one, which was defined in terms of its properties. These advances allowed the solution of 4rd and 5rd order equations. A student of Cardano, for example, Rafael Bombelle in 1572 wrote: "*meno di meno via meno di meno fa meno*" which in our, cartesian, terms would correspond to  $(-i) \times (-i) = -1$  [which in a modern type definition corresponds to an operational statement of a (new) number  $i$  and in fact to the birth of a new algebra] which then was not related to a geometrical concept although it could, from our actual point of view, be considered as generating the negative side of a surface of unit area.

The French school also started with explicit geometric concepts. François Viète (1540-1603), for example, wrote equations where volumes are explicitly considered:

"B in A Quadratum, plus D plano aequari Solidos".

Let's remember for example the work of Pierre de Fermat (1601-1665) where he clearly relates the quadratic curves to lines generated by plane cross sections of cones.

Of course modern algebra started with René Descartes (1596-1650) where not only our point of view of algebra, a clear adoption of Persian numerals and our actual algebraic notation was started, but also clear concepts as the need of a reference scalar (consider for example that his equation  $e : a = b : c$

corresponds to our  $ab = c$  when the value of  $e$  has been fixed as unity and that he understands  $c$  as an area  $ce$  while  $a, b, c, \dots$  correspond to line segments. The fact that he used expressions like  $a + b, a - b, ab, a/b, \sqrt{a}, \dots$  marked not only our notation but the possibility of abstract work where the correspondence of variables to actual physical quantities was not longer needed. The use of the cartesian plane also marked our line of thoughts whenever discussing algebra and its relations to geometry. He explicitly had in mind the algebra generated by numbers on directed lines.

It was also Descartes which extended the operational concept of complex numbers introduced by Cardano and Bombelli (and the criticism of authors like Albert Girard which, by calling complex solutions “impossible solutions”, discarded complex numbers from a possibility of real existence) by calling those solutions “imaginary” and using phrases which were not meant to discard them or to consider them useless but to give them an acceptable existence: “*imaginer tant de solutions...*”. A century later Leonhard Euler gave full citizenship to complex solutions by a systematic use of them.

Instead of going on with a detailed description of the developments on this subject in the last two centuries we will rather start by a quick overview of the evolution of geometric numbers. We leave this analysis, and considerations about vector, differential forms and tensor calculus, for a future comment. (See the article of Prof. Parra in this issue, for some historical and methodological considerations). Caspar Wessel (1797) introduces directed numbers on a tridimensional basis set. Jean Robert Argand (1806) discusses the complex “plane”. John Warren (1828) and Carl Friedrich Gauss (1831) find properties of the complex plane. The great precursor Herman Günther Grassman (1844) creates n-dimensional vector spaces and multivectors. While William Rowan Hamilton (1853) creates a new algebra useful for three dimensional space, quaternions, uses it extensively and convinces the mathematical world of its usefulness. William Kingdom Clifford (1873) creates the associative multivector algebra which now bears his name. Arthur Cayley creates two abstract algebras that of matrices (1858) and that of groups (1854) which we know now are homomorphic to multivector algebras, he also showed (1885) that quaternions could be used to generate three-and four-dimensional rotations.

O. Heaviside (1885) and John Williard Gibbs (1905) over emphasized vectors, and a conceptual step backwards is in fact given although, as a result of its appealing simplicity, vector algebra as is now widely used is born, with its unnecessary factual complexity. We must of course remind the reader of the birth of a new mathematical entity: the spinors, with the work of Elie Cartan (1917), (1938), Wolfgang Pauli (1923), Paul Adrian Maurice Dirac (1928) and the seminal work of Richard Brauer and Hermann Weyl (1935) or its modern conception as minimal left (right) ideals of Clifford algebras by Marcel Riesz (1943).

All these developments clearly show that a name “geometric algebra” is undoubtedly correct but that in fact, for the learned community, should really be a pleonasm because algebra has always been geometric. On the other hand abstract formalisms are very important to consider as such, deprived of a physical anchor, for the purpose of freedom in their conception and use.

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Extracted from a talk presented at the Centro de Fijación de Nitrógeno, U.N.A.M., within the series “Beyond Nitrogen Fixation”, May 4th, 1992.

