ALGEBRAIC AND DIFFERENTIAL EQUATIONS FOR SPINNING PARTICLES ON THE SPHERE

Jaime Keller*
División de Estudios de Posgrado, Facultad de Química, and
Facultad de Estudios Superiores–Cuautitlán
Universidad Nacional Autónoma de México
A. Postal 70-528, 04510 México, D. F., México
e-mail: keller@servidor.unam.mx

Robert M. Yamaleev
Facultad de Estudios Superiores–Cuautitlán
Universidad Nacional Autónoma de México
On leave: Joint Institut for Nuclear Research, Dubna, Russia
e-mail: robert@nutrius.cuautitlan1.unam.mx

and

Adán Rodríguez
Instituto de Física: "Manuel Sandoval Vallarta"
Universidad Autónoma de San Luis Potosí,
A. Postal 2–22, 78216–San Luis Potosí, México
e-mail: adnrdz@dec1.ifisica.uaslp.mx

(Received: June 8, 1998; Accepted: August 5, 1998)

Abstract. We revise the mathematical formulation of the theory of a particle in a spherical surface, in particular we show that the system of relations between two sets of generators of the $SU(2)$ group lead to a formulation of nonrelativistic spin-one half theory on the sphere S^3. First we examine various possibilities to extend this approach in the case of relativistic motion, then we give formulation for the Dirac and Maxwell equations in homogeneous space-time where a geometrical point is associated with the notion of relativistic top. Finally we formulate these equations in a S^3 surface embedded in R^5, using spherical system of coordinates, and examine the eigenvalue problem.

* Author to whom all correspondence should be addressed

Advances in Applied Clifford Algebras **8** No. 2, 235-254 (1998)

1. Introduction

Among all possible values s the spin has a special status. It is frequently considered that spin-one half is the basic spin serving as building block for systems with other values of spin. Currently it has been well established that the theory of spin-one half is closely connected with the theory of space-time. This point of view has been developed, for example, in the theory of spinor representation of the space -time [1, 2, 3, 4] or phase space coordinates (twistors) [5, 6, 7, 8, 9] and in the description of space- time within the basis of Clifford algebras [10, 11, 12, 13, 14]. It is important that all these approaches lead to a geometrical basis beyond the standard frames for space-time. It is also important to recognize that the Clifford algebra approach faithfully and completely embeds the spinor and twistor formulations.

In this paper we display the deep relation between the set of generators of the space-time group and the system of equations for spin-one half particles.

2. Nonrelativistic Equation of Motion for a Particle with Spin One Half in the Basis of Generators of the Group $SU(2)$

We start with a simple example to display the relation between the set of relationships of the generators of the Galilean group and the structure of the Hamiltonian of a particle with spin 1. At the level of mechanics the relation between a free particle of mass m Hamiltonian and momentum p is given by

$$H = \frac{1}{2m}\,\vec{p}^{\,2}. \tag{2.1}$$

This form of the Hamiltonian plays a fundamental role in the classical and quantum mechanics, in particularly, it determines the structure of the Hamilton-Jacobi and Schrödinger equations. Within quantum mechanics the algebraic relation (2.1) is understood as a relation between operators for momentum and energy given by

$$H = i\hbar\frac{\partial}{\partial t}, \quad \vec{p} = -i\hbar\frac{\partial}{\partial \vec{x}}. \tag{2.2}$$

This is a formal way to obtain the Schrödinger equation for the nonrelativistic spinless particles. Operators H and p are generators of the Galilean group. Now it comes the question: is it possible to generalize the relation (2.1) to obtain the quantum equations for a particles with a spin? One, straightforward, form is to consider the Clifford algebra for space (Pauli complex algebra) [11b]. It turns out that this program may be realized by making use of the relationships between the generators of the Galilean group. These relationships are given by the following simple system (here $(\vec{a},\vec{b})$ denotes the symmetric scalar product between vectors $\vec{a}$ and $\vec{b}$

$$\vec{M}=[\vec{r}\times\vec{p}],\quad \vec{r}\,2mH=-[\vec{M}\times\vec{p}]+s\vec{p},\quad \text{where}\quad s=(\vec{r},\vec{p}). \tag{2.3}$$

The determinant of this system equals zero, yielding:

$$2mH-\vec{p}^{\,2}=0. \tag{2.4}$$

An important property of (2.3) is observed when the particle is considered to move in an external electromagnetic field. Let us introduce the field potentials $\vec{A}$ and ϕ in a canonical way:

$$\vec{p}\rightarrow\vec{p}'=\vec{p}+\frac{e}{c}\vec{A},\quad H\rightarrow H-e\phi. \tag{2.5}$$

In that case the components of the momentum operator don't commute. Moreover the commutator is

$$[\vec{p}^{\,\prime}\times\vec{p}^{\,\prime}]=-ih\frac{e}{c}\vec{\mathcal{H}}, \tag{2.6}$$

where $\vec{\mathcal{H}}$ is the bivector of the magnetic field's strength. The system (2.3) contains then the expression for the Hamiltonian considering spin in a hidden way. Let us introduce the 3-dimensional orthonormal basis $\{\vec{e}_k\}$, $k=1,2,3$ $(\vec{e}_j,\vec{e}_i)=\delta_{ji}$. In this paper we have used the symbol $(\vec{a},\vec{b})$ to denote either the scalar product $d=(\vec{a},\vec{b})=a^ib_i$ or, as in (2.8) below, the formulation of a vector in Clifford algebra form $\mathbf{A}=(\vec{a},\vec{\delta})=a^\mu\delta_\mu$ where $\{\delta_\mu\}$ is a basis vector set in Clifford algebra. Also the three dimensional map $\vec{\tau}_{ik}=[e_i\times e_k]=e_{ik}$. Within this basis the system (2.3) becomes

$$\{\vec{r}\}\;2mH=\{\vec{r}\}\{\vec{p}^{\,2}+(\vec{\tau},[\vec{p}\times\vec{p}])\}, \tag{2.7}$$

where the matrix-vector $\vec{\tau}_{ik}$ is given by

$$\vec{\tau}_{ik}=[\vec{e}_i\times\vec{e}_k].$$

The last term of (2.7) corresponds to energy of the interaction between the magnetic momentum of the spin $\vec{S}=-i\hbar\vec{\tau}$ and the magnetic field . In fact,

$$\tfrac{1}{2m}(\vec{\tau},[\vec{p}\times\vec{p}])=-ih\tfrac{e}{2mc}(\vec{\tau},\vec{H})=\tfrac{e}{2mc}(\vec{S},\vec{H}).$$

Then the spin-one matrices can be constructed from the orthonormal basis vectors of the three-dimensional space. Now let us try the same procedure to obtain the Hamiltonian of spin-one half particles. In the standard approach the spin-one half-operator is represented by Pauli matrices $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$, we then rewrite the system (2.3) in the following way

$$(\vec{r},\vec{\sigma})(2mH) = (s + (\vec{M},\vec{\sigma}))(\vec{p},\vec{\sigma}),$$

$$s + (\vec{M},\vec{\sigma}) = (\vec{r},\vec{\sigma})(\vec{p},\vec{\sigma}). \tag{2.8}$$

We note, however, that these algebraic relations may be fulfilled iff

$$(\vec{M},\vec{p}) = (\vec{r},[\vec{p}\times\vec{p}]) = 0, \tag{2.9}$$

which obviously should be obeyed for free particles, because

$$[\vec{p}\times\vec{p}] = 0.$$

But this is not the case in the presence of an external electromagnetic field. From (2.6) one gets

$$(\vec{M},\vec{p}) = -ih\frac{e}{c}(\vec{r},\vec{\mathcal{H}}) = 0. \tag{2.10}$$

Obviously this equality could not be satisfied in the general case because one may choose the origin of the reference system by arbitrary way. The cause of this contradiction lies in the non-completeness of the algebraic basis $\{\sigma_1,\ \sigma_2,\ \sigma_3\}$ which we have used. Indeed the condition (2.9) disappears as soon as we use the complete ring as a basis $\{I,\ \sigma_1,\ \sigma_2,\ \sigma_3\}$. This is a basis of a four -dimensional Euclidean space. Correspondingly the system (2.8) takes the form

$$\{r_4 + i(\vec{r},\vec{\sigma})\}\ 2mH = \{s + i(\vec{M}+\vec{N},\vec{\sigma})\}\{p_4 + (\vec{p},\vec{\sigma})\}, \tag{2.11}$$

$$s + i(\vec{M}+\vec{N},\vec{\sigma}) = \{r_4 + i(\vec{r},\vec{\sigma})\}\{p_4 - i(\vec{p},\vec{\sigma})\},$$

where $\vec{N} = r_4\vec{p} - \vec{r}p_4,\ s = (\vec{r},\vec{p}) + r_4p_4$. Separating the expressions for the basic units $\{I,\ (\sigma_1,\ \sigma_2,\ \sigma_3)\}$ we obtain

$$r_4\ 2mH = ((\vec{M}+\vec{N}),\vec{p}) + s\ p_4,$$

$$\vec{r}\ 2mH = -[(\vec{M}+\vec{N})\times\vec{p}] + s\vec{p} - (\vec{M}+\vec{N})\ p_4, \tag{2.12}$$

which contains the Hamiltonian

$$2mH = p_4^2 + \vec{p}^{\ 2} + (\vec{\tau}_+,([\vec{p}\times\vec{p}] + (p_4\vec{p} - \vec{p}\ p_4)). \tag{2.13}$$

In the left handed system of reference we get

$$r_4\ 2mH = ((\vec{M}-\vec{N}),\vec{p}) + s\ p_4,$$

$$\vec{r}\ 2mH = -[(\vec{M}-\vec{N})\times\vec{p}] + s\vec{p} - (\vec{M}-\vec{N})\ p_4,$$

with Hamiltonian

$$2mH = p_4^2 + \vec{p}^{\ 2} + (\vec{\tau}_-, ([\vec{p}\times\vec{p}] - (p_4\vec{p} - \vec{p}\ p_4)), \tag{2.14}$$

the spin-matrix being given by

$$\vec{\tau}_{\pm(\alpha\beta)} = [\vec{e}_\alpha \times \vec{e}_\beta] \pm (e^4_\alpha \vec{e}_\beta - e^4_\beta \vec{e}_\alpha), \tag{2.15}$$

with $\{\vec{e}_k\}$, $k = 1, 2, 3, 4$ and $(\vec{e}_i, \vec{e}_j) = \delta_{ij}$.

Let us note the operators $\vec{N}$, $\vec{M}$ are generators of the $SO(4)$ group while the linear combinations of these operators

$$\vec{\mathcal{M}}_\pm/2 = (\vec{M} \pm \vec{N})/2 \tag{2.16}$$

form two set of generators for the $SU(2)$ group. The operator $s = (\vec{r},\vec{p}) + r_4 p_4$ is the dilatation generator.

The isomorphism between the $SO(4)$ and $SU(2)$ groups is explicitly exhibited in the vector-parametrization of the $SO(4)$ group by

$$T[SO(4)] = \frac{(1+\hat{a}_+)(1+\hat{b}_-)}{\sqrt{(1+\vec{a}^2)(1+\vec{b}^2)}} = T_+(\vec{a})T_-(\vec{b}).$$

In this representation the spin-matrices (2.15) are interpreted as matrices of infinitesimal transformations of the $SU(2)$ group:

$$\vec{\tau}_+ = \partial T_+/\partial\vec{a}(\vec{a}=0), \quad \vec{\tau}_- = \partial T_-/\partial\vec{b}(\vec{b}=0).$$

And in this formulation the spin-one half operator is defined by

$$\vec{S} = -i\hbar\vec{\tau}_\pm/2,$$

instead of the usual one given in terms of the Pauli matrices

$$S = \tfrac{\hbar}{2}\vec{\sigma}.$$

The basis $\{I, -i\tau_x, -i\tau_y, -i\tau_z\}$ is then isomorphic to the basis of Pauli matrices $\{I, \sigma_x, \sigma_y, \sigma_z\}$.

Our purpose is to formulate the Pauli equation purely in the terms of the generators of the $SU(2)$ group or, in other words, in terms of the operators σ and $\vec{M}$.

The Hamiltonian of the spin-one half particle in 4-dimensional Euclidean space is written as follows

$$2m\ H = (p_4 - i(\vec{p},\vec{\sigma}))(p_4 + i(\vec{p},\vec{\sigma})). \tag{2.17}$$

Now let us use the constraint for four-dimensional coordinates to a sphere

$$R^2 = r_4^2 + \vec{r}^{\ 2} = (r_4 + i(\vec{r},\vec{\sigma}))(r_4 - i(\vec{p},\vec{\sigma})), \tag{2.18}$$

where R is the radius of the three-dimensional sphere S^3. We get

$$2m\ H = \frac{1}{R^2}\{p_4 - i(\vec{p},\vec{\sigma})\}\{r_4 + i(\vec{r},\vec{\sigma})\}\{r_4 - i(\vec{r},\vec{\sigma})\}\{p_4 + i(\vec{p},\vec{\sigma})\} = \tag{2.19}$$

$$(s' - i(\vec{\mathcal{M}},\vec{\sigma}))(s + i(\vec{\mathcal{M}},\vec{\sigma}),$$

where $s = (\vec{r},\vec{p}) + r_4 p_4$ is the Euler operator with the eigenvalue $-4i\hbar(0, 1, 2, ...)$. On the sphere S^3 one of the eigenvalues of this operator is zero. The eigenvalue of the operator $s' = (\vec{p},\vec{r}) + p_4 r_4$ is $-4i\hbar$. Restricting ourselves to this eigenvalue of s only we obtain

$$2m\ H = \frac{i}{R^2}\{-4i\hbar - i(\vec{\mathcal{M}},\vec{\sigma})\}(\vec{\mathcal{M}},\vec{\sigma}). \tag{2.20}$$

To find the eigenvalues and eigenfunctions of the Hamiltonian (2.20) we use the expansion of the solution in spherical harmonics

$$Y_{JM}^{LS}(\theta,\phi,\psi) = \textstyle\sum_{m,\sigma} C_{LmS\sigma}^{JM} Y_{Lm}(\theta,\phi,\psi)\xi_{S\sigma}.$$

where $(\vec{\mathcal{M}} + \vec{S})^2 Y_{JM}^{LS} = J(J+1)Y_{JM}^{LS}$ and $(\vec{S})^2 Y_{JM}^{LS} = \frac{3}{4}Y_{JM}^{LS}$, to obtain

$$(\vec{\mathcal{M}}^2 + 2(\vec{S},\vec{\mathcal{M}})\}Y_{JM}^{LS} = (J(J+1) - \tfrac{3}{4})Y_{JM}^{LS},$$

with $J = 0, \frac{1}{2}, 1, \frac{3}{2}, ...$. Let us first concentrate our attention on the ground state $J = 0$. The eigenvalue corresponding to the ground state is

$$H(J=0) = -\tfrac{2}{mR^2}\vec{S}^2 \rightarrow -\tfrac{\hbar^2}{mR^2}\tfrac{3}{2},$$

a negative value. To avoid negative values in the <u>definition</u> of the energy we define the Hamiltonian as the sum

$$H = H + \tfrac{2}{mR^2}\vec{S}^2 = \tfrac{2}{mR^2}\{\vec{\mathcal{M}}^2 + 2(\vec{S},\vec{\mathcal{M}})\} + \tfrac{2}{mR^2}\vec{S}^2.$$

The term $\frac{2}{mR^2}\vec{S}^2$ we can interpret as a reference energy corresponding to the spin of the particle. We then get a remarkable expression for the Hamiltonian of a spin-one half particle, on the sphere S^3, as follows

$$H = \frac{2}{mR^2}(\vec{\mathcal{M}}^2 + \vec{S})^2. \tag{2.21}$$

The spectrum of this operator is immediately given by

$$H\Psi(J) = \tfrac{2\hbar}{mR^2} J(J+1)\Psi(J), \quad J = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots .$$

The discreteness of the energy spectrum is a consequence of the compactness of the group $SU(2)$, the space of which is the space of the solutions. When $R \to \infty$ the Hamiltonian (2.21) tends to the Hamiltonian of the free particle Pauli equation. In that case

$$\vec{\mathcal{M}}_{\pm}/R = (\vec{M} \pm \vec{N})/R \to \pm\vec{p},$$

and

$$H = \frac{2}{mR^2}(\frac{\vec{M}}{2} + \vec{S})^2 = \frac{1}{2mR^2}((2\hbar - i\vec{\tau}\vec{\mathcal{M}})^2 - \hbar^2) \to \frac{1}{2m}(\vec{p}, \vec{\sigma})^2.$$

As it has been shown in [15], the Hamiltonian (2.21) corresponds at the classical level to a spherical symmetric classical top's Hamiltonian on S^3:

$$H = \tfrac{J^2}{2I},$$

where I is a moment of inertia.

Let us consider the motion constrained to S^3 and in an external magnetic field. As an antecedent the nonrelativistic equation for spinless particles in spherical space had first been formulated by Schrödinger [16]. Second the Pauli equation considers generalized momentum in the gauge invariant form

$$\vec{P} = \vec{p} + \tfrac{e}{c}\vec{A},$$

to get, when Clifford algebra is considered $(\vec{p}, \vec{\delta}) = p^{\mu}\delta_{\mu}$, the additional quantity in the Hamiltonian responsible of the interaction between the magnetic momentum of the charged spin particle with the magnetic field. This quantity "automatically" appeared in the Hamiltonian as soon as we include an external magnetic field in the gauge invariant formulation

$$2mH = (\vec{P}, \vec{\sigma})(\vec{P}, \vec{\sigma}) = \vec{P}^2 + ([\vec{P} \times \vec{P}], \vec{\sigma}) = (\vec{p} + \tfrac{e}{c}\vec{A})^2 + \hbar\tfrac{e}{c}(\vec{\mathcal{H}}, \vec{\sigma}),$$

where

$$\vec{\mathcal{H}} = \tfrac{ic}{e\hbar}[(\vec{p} + \tfrac{e}{c}\vec{A}) \times (\vec{p} + \tfrac{e}{c}\vec{A})].$$

On the hypersphere S^3 as a result of the introduction of the gauge field we get [17]

$$\vec{M} = \vec{M} + \tfrac{e}{c}\vec{B},$$

where the constrained gradient operator $\vec{B} = \frac{1}{R}\{[\vec{r}\times\vec{A}] + (R\vec{A} + \frac{\vec{r}}{R}(\vec{r},\vec{A}))\}$. The strength of the magnetic field is given by

$$\vec{G} = \frac{c}{e}\{[(\vec{\nabla} + \frac{e}{c}\vec{B}) \times (\vec{\nabla} + \frac{e}{c}\vec{B})] - \frac{2}{R}(\vec{\nabla} + \frac{e}{c}\vec{B})\} =$$

$$= ([\vec{\nabla}\times\vec{B}] - \tfrac{2}{R}\vec{B})),$$

where the constrained gradient operator $\vec{\nabla} = \frac{1}{R}\{[\vec{r}\times\frac{\partial}{\partial\vec{r}}] + (R\frac{\partial}{\partial\vec{r}} + \frac{\vec{r}}{R}(\vec{r},\frac{\partial}{\partial\vec{r}}))\}$, whose components obey the commutation relations

$$[\nabla_i, \nabla_j] = 2e_{ijk}\nabla_k, \quad i,j,k = 1,2,3.$$

3. Extension of Space-time Coordinates Conserving the Poincaré Group Relations

We generalize the relevant formulae from the previous section to the relativistic case. First all the basic operators considered have to belong to the Poincaré group. The generators of this group are the 4-components of momentum p^a and the 6-components of the angular momentum M^{ab}, from which one builds two Casimir operators (indices $a, b, c, d, \ldots$ run from 1 to 4)

$$m^2 = p^a p_a, \quad W^2 = W^a W_a, \quad W_a = -\frac{1}{2}e_{abcd}M^{bc}p^d, \tag{3.1}$$

where W_a is the Pauli-Lubanski (axial) vector, which is supposed to be proportional to the total spin of the spinning system. In fact p is a vector, M a bivector and W a trivector mapped, by duality, into a vector like quantity. When the system is isolated, p^a and M^{ab} are constants of motion and for any positive m, the history of the center-of-mass consists of the points of M:

$$m^2\ x^a = M^{ab}p_b + sp_a, \quad s = x^a p_a, \quad M^{ab} = (x^a p^b - x^b p^a). \tag{3.2}$$

The condition for non-triviality of the solutions of this system is given by

$$m^2 - p^a p_a = 0.$$

The set of relations (3.2) generates the well known Proca equations [18] which can be written as:

$$m^2\ u^a = -p_b U^{ab} + p^a U_0, \quad U_0 = p_a u^a, \quad U^{ab} = (p^a u^b - p^b u^a). \tag{3.3}$$

Thus we have generalized the system (2.3) to the relativistic case which again occurs related to spin-one equations. The question arises: which kind of structural relations must be used for the generators of the Poincaré group to obtain the relativistic equations of motion for spin-one half particles? First let us note that in the system of relations (3.2) the Pauli-Lubanski operator, which plays essential role in the formulation of Poincaré groups for spinning particles, disappeared because the Pauli-Lubanski vector is trivial for the angular momentum taken as orbital angular momentum. It is well known that its value in non-trivial iff the angular momentum is taken as total angular momentum consisting of orbital and spin parts. Our goal is to define the spin part by extending the coordinate part in the relations (3.2). For that purpose let us consider first the massless case where the extension of the coordinate part yields $W_a = sp_a$.

For our purpose we present the spin part in the form

$$S^{ab} = L^{abc}p_c = \frac{1}{2}e^{abcd}y_c p_d, \tag{3.4}$$

with

$$L^{abc} = \frac{1}{2}e^{abcd}y_d, \tag{3.5}$$

where L^{abc} corresponds to a trivector therefore it is skew symmetric in the tensor indices. Inserting the total angular momentum defined by

$$M^{ab} = (x^a p^b - x^b p^a) + L^{abc}p_c, \tag{3.6}$$

into (3.1) we get

$$W_a = -\frac{1}{2}e_{abcd}p^b e^{cdpq}y_p p_q = -p_a(y^b p_b) = sp_a, \tag{3.7}$$

with $(y^b p_b) = -s$. Thus for $m = 0$ we can build the following identities

$$M^{ab}p^b + (x_b p^b)p_a = 0,$$

$$\frac{1}{2}\eta_{abcd}M^{bc}P^d + sp_a = 0, \quad (y^b p_b) = -s. \tag{3.8}$$

We suggest that the spin part should have also the same structure for the case of positive mass, so that the total angular momentum is defined by (3.6). From the L^{abc} and p_a one can build also the fully skew symmetric tensor K^{abcd}:

$$K^{abcd} = L^{[abc}p^{d]} = \frac{1}{4}(L^{abc}p^d + L^{bcd}p^a + L^{cda}p^b + L^{dab}p^c). \tag{3.9}$$

In four -dimensional space this tensor is proportional to the also fully antisymmetric Ricci tensor e^{abcd} with entries 1 or -1 according to the parity of the permutations of *abcd*:

$$K^{abcd} = \frac{1}{4} e^{abcd} K_0. \tag{3.10}$$

One may also invert this relation getting the contraction

$$K_0 = \frac{1}{6} e_{abcd} K^{abcd}. \tag{3.11}$$

With those definitions one may obtain the following identity

$$m^2 \ L^{abc} = 3p^{[a} M^{bc]} + 4p_d K^{abcd}, \tag{3.12}$$

where $3p^{[a}M^{bc]} = p_a M^{bc} + p_b M^{ca} + p_c M^{ab}$. Remembering that the tensor L^{abc} is linked to an axial four- vector by (3.5) we can be transform (3.12) into the following system

$$m^2 \ y^a = \frac{1}{2} e^{abcd} M_{bc} p_d + K_0 p^a. \tag{3.13}$$

Now let us summarize all these identities into one set of relations. We get

$$m^2 \ x^a = M^{ab} p_b + s p_a, \quad s = x^a p_a, \quad M^{ab} = (x^a p^b - x^b p^a) + L^{abc} p_c,$$

$$m^2 \ L^{abc} = 3p^{[a} M^{bc]} + 4p_d K^{abcd}, \ K^{abcd} = L^{[abc} p^{d]}. \tag{3.14}$$

This set of relations generates the well known Dirac-Kähler equations [19]. By dividing M_{ab} into self-dual and anti-self dual parts:

$$M_{ab} = M^-_{ab} + M^+_{ab}, \quad M^+_{ab} = \tfrac{1}{2} e^{cd}_{ab} M_{cd} = +iM^-_{ab}$$

and introducing the complex values

$$x^{+a} = x^a + iy^a, \ x^{-a} = x^a - iy^a,$$

$$w^+ = s + iK_0, \ w^- = s - iK_0,$$

we can transform (3.14) as follows

$$\begin{aligned} m^2 \ x^{+a} &= M^-_{ab} p_b + w^- p^a, \\ m^2 \ x^{-a} &= M^+_{ab} p_b + w^+ p^a. \end{aligned} \tag{3.15}$$

This system of operators generates the following equations of motion [20]

$$\partial^\sigma \psi_\sigma = \tfrac{mc}{\hbar} \psi,$$

$$\partial^\mu \psi_\nu - \partial^\nu \psi_\mu - i e_{\mu\nu\alpha\beta} \partial^\alpha \psi^\beta = \tfrac{mc}{\hbar} \psi_{\nu\mu},$$

$$\partial^\sigma \psi_{\sigma\mu} - \partial_\mu \psi = \tfrac{mc}{\hbar} \psi_\mu.$$

4. Generalization to the Relativistic Case

Following Pestov [21] we will consider a homogeneous space-time that differs from the Minkowski space-time by geometrical and topological properties. We remark here that in Clifford algebra this is equivalent to use complex algebra and some constraint [14b]. In the non-Euclidean five-dimensional Minkowski space-time M^5_{14} with Cartesian coordinates x^λ , $\lambda = 0, 1, 2, 3, 4$

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 - (dx^4)^2 \tag{4.1}$$

consider the constrain to the one sheet hyperboloid H^4 given by

$$\eta_{\lambda\mu} x^\lambda x^\mu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2 = -R^2, \tag{4.2}$$

where R is the hyperbolic radius of H^4. The vector fields

$$P_\lambda = \delta^\mu_\lambda \partial_\mu, \quad M_{\lambda\mu} = (x_\lambda \delta^\nu_\mu - x_\mu \delta^\nu_\lambda)\partial_\nu, \tag{4.3}$$

where $x_\lambda = \eta_{\lambda\mu} x^\mu$, are generators of the Poincaré group of the five-dimensional Minkowski space-time. All vector fields $M_{\lambda\mu}$ are orthogonal to the radius-vector $R = x^\nu \partial_\nu$. Using this expand P_ν in the direction of the radius-vector R and the one orthogonal to it, to obtain the vector fields

$$M_\mu = aP_\mu + \frac{1}{a}(R, P_\mu), \tag{4.4}$$

The vector fields M_μ and $M_{\mu\nu}$ are generators of the group of conformal transformations of H^4 because

$$[M_\mu, M_\nu] = -M_{\mu\nu}, \quad [M_\lambda, M_{\mu\nu}] = \eta_{\lambda\mu} M_\nu - \eta_{\lambda\nu} M_\nu. \tag{4.5}$$

Define now the vector fields (compare with (2.16))

$$\mathcal{M}_0 = M_0, \quad \mathcal{M}_1 = M_{14} + M_{23}, \quad \mathcal{M}_2 = M_{24} + M_{31}, \quad \mathcal{M}_3 = M_{34} + M_{12}, \tag{4.6}$$

with components

$$\begin{aligned} \mathcal{M}_0 &= (R + \tfrac{1}{2}x_0^2, \tfrac{1}{2}x_0x^1, \tfrac{1}{2}x_0x^2, \tfrac{1}{2}x_0x^3, \tfrac{1}{2}x_0x^4), \\ \mathcal{M}_1 &= (0, -x_4, -x_3, x_2, x_1), \\ \mathcal{M}_2 &= (0, x_3, -x_4, -x_1, x_2), \\ \mathcal{M}_3 &= (0, -x_2, x_1, -x_4, x_3). \end{aligned} \tag{4.7}$$

The vector fields $\mathcal{M}_0,\ \mathcal{M}_l,\ \mathcal{M}_2,\ \mathcal{M}_3$ are continuous, linearly independent and do not vanish at any point of H^4. They obey $(\mathcal{M}_a, \mathcal{M}_b) = 0$ for $a \neq b,\ a, b = 0, 1, 2, 3$ and

$$(\mathcal{M}_0, \mathcal{M}_0) = -(\mathcal{M}_1, \mathcal{M}_1) = -(\mathcal{M}_2, \mathcal{M}_2) = -(\mathcal{M}_3, \mathcal{M}_3) = R^2 + x_0^2.$$

From (4.5)-(4.7) follow the commutation relations

$$[\mathcal{M}_0, \mathcal{M}_i] = 0, \quad \text{and} \quad [\mathcal{M}_i, \mathcal{M}_j] = 2e_{ijk}\mathcal{M}_k, \quad i, j, k = 1, 2, 3. \tag{4.8}$$

The one sheet hyperboloid (4.2) admits a simply transitive group of transformations with the generators (4.6) having as the only nontrivial structure constants

$$f_{23}^1 = f_{31}^2 = f_{12}^3 = 2. \tag{4.9}$$

Therefore, we will supply H^4 with a metric of the type (4.9) and thus transform H^4 into the hyperbolic space-time H_{13}^4. From (4.6)-(4.9) it follows that the vector field $\mathcal{M}_0$ is absolutely parallel with respect to the connection on H_{13}^4 induced by the vector fields (4.6). For comparison we note that the vector field $\mathcal{M}_0 = \frac{\partial}{\partial x^0}$ defined in (4.7) is also absolutely parallel. Also it can be shown that in the homogeneous space-time H_{13}^4 a geometrical point of the spatial cross section can be associated with the notion of a top. From (4.2)-(4.9) it follows that the cross section of H^4 at the hyperplane $x^0 = 0$ is a three-dimensional spherical surface

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = R^2 \tag{4.10}$$

immersed in a four-dimensional Euclidean space.

Now within these geometrical notions we can generalize the equation (2.21) for the top. Let us remember before the generalization of the Pauli equation into the Dirac equation. This can be displayed in the following scheme:

$$H_P = \frac{1}{2m}(\vec{p}, \vec{\sigma})^2 \to (\tfrac{H}{c} - mc)(\tfrac{H}{c} + mc) = (\vec{p}, \vec{\sigma})^2 \to$$

$$\to Det \begin{pmatrix} \frac{H_D}{c} - mc & (\vec{p}, \vec{\sigma}) \\ (\vec{p}, \vec{\sigma}) & \frac{H_D}{c} + mc \end{pmatrix} = 0 \to$$

$$\to \tfrac{H_D}{c}\Psi = (\vec{\alpha}, \vec{p}) + \beta mc)\Psi.$$

In the same way we get [22]

$$2mH = \tfrac{((\vec{\sigma},\vec{\mathcal{M}})+2\hbar)}{R}\tfrac{((\vec{\sigma},\vec{\mathcal{M}})+2\hbar)}{R} \to$$

$$\to (\tfrac{H_D}{c} - mc)(\tfrac{H_D}{c} + mc) = ((\vec{\sigma},\vec{\mathcal{M}}) + 2\hbar)^2 \to$$

$$\to Det \begin{pmatrix} \frac{H_D}{c} - mc & (\vec{\sigma},\vec{\mathcal{M}}) + 2\hbar \\ (\vec{\sigma},\vec{\mathcal{M}}) + 2\hbar & \frac{H_D}{c} + mc \end{pmatrix} = 0 \to$$

$$\to \frac{H_D}{c}\Psi_\pm = \left((\vec{\alpha},\ \frac{\vec{\mathcal{M}}_\pm}{R}) + \beta mc + \gamma_5\,\frac{2\hbar}{R} \right)\Psi_\pm, \quad \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{4.11}$$

The spectrum of this equation for the free motion case may easily be found taking into account that the operator $(\vec{\sigma},\vec{\mathcal{M}}) + 2\hbar$ commute with H_D. The eigenvalues of this operator are known

$$[(\vec{\sigma},\vec{\mathcal{M}}) + 2\hbar]\Psi = (n+1)\Psi, \quad n = 0, 1, 2, \ldots .$$

As a most important result we find following formula for the spectrum of (4.11)

$$\mathcal{E} = c\sqrt{m^2c^2 + \frac{\hbar^2(n+1)^2}{R^2}}. \tag{4.12}$$

From physical considerations it is clear that for large n and R, the formula (4.12) will approach the properties of the classical top. Indeed, in the limit of large R it follows from (4.12) that

$$\mathcal{E} = mc^2 + \tfrac{J^2}{2I}.$$

The Dirac's equation was first generalized to the de-Sitter space by Dirac itself [23]. He also formulated this equation in the conformal space [24].

Now let us consider the Dirac—Maxwell system of equations in the hyperbolic space-time. We write the Dirac equation in the homogeneous space-time in the form

$$\gamma^c D_c \psi = \mu\psi, \tag{4.13}$$

where the Minkowski space Clifford algebra is generated by the well known γ^α

$$\gamma^a\gamma^b + \gamma^b\gamma^a = -2\eta^{ab}, \quad \text{and} \quad D_a = \nabla_a + \tfrac{ie}{\hbar c}B_a - \tfrac{1}{2}f^b_{ba},$$

the B_a being the components of the vector potential of the electromagnetic field. Taking into account the structure constants (4.9) $[\nabla_a, \nabla_b] = f^c_{ab}\nabla_c$, we have

$$[D_a, D_b] = f^c_{ab}D_c + \tfrac{ie}{\hbar c}G_{ab},$$

where

$$G_{ab} = \nabla_a \nabla_b - \nabla_b B_a - f^c_{ab} B_c$$

are the components of the strength tensor of the electromagnetic field in the basis ∇_a. The Jacobi identity $[D_a[D_b, D_c]] + [D_b[D_c, D_a]] + [D_c[D_a, D_b]] = 0$ results in the first four Maxwell equations

$$\nabla_a G_{bc} + \nabla_b G_{ca} + \nabla_c G_{ab} + f^d_{ab} G_{cd} + f^d_{bc} G_{ad} + f^d_{ca} G_{bd} = 0. \tag{4.14}$$

To establish the other four Maxwell equations we use

$$\widetilde{G}^{ab} = \tfrac{1}{2} e^{abcd} G_{cd}.$$

Then eq. (4.14) can be written in the following equivalent form

$$\nabla_a \tilde{G}^{ab} + f_a \tilde{G}^{ab} + \frac{1}{2} f^b_{ad} \tilde{G}^{ad} = 0, \tag{4.15}$$

from which follows the second group of Maxwell equations

$$\nabla_a G^{ab} + f_a G^{ab} + \frac{1}{2} f^b_{ad} G^{ad} = \frac{4\pi a}{c} j^b, \tag{4.16}$$

where the j^b are components of the current vector. In three-dimensional vector form the Maxwell equations are written

$$j^a = (c\rho, \vec{j}), \quad A_a = (\phi, -\vec{A}), \quad E_i = F_{0i}, \quad H_i = \tfrac{1}{2} e_{ijk} F^{jk}, \quad i, j, k = 1, 2, 3.$$

And from (4.14)- (4.16) we obtain the standard formulae

$$\vec{E} = -\nabla_0 \vec{A} - \nabla \phi, \quad \vec{H} = rot \vec{A} = \nabla \times \vec{A} - 2\vec{A}. \tag{4.17}$$

Now consider the Coulomb law in equations (4.14)-(4.16). In Euclidean space the Coulomb potential can be derived as a solution of the equations of electrostatics, invariant under the Galilean group of motions including rotations and translations. We then look for the Coulomb-like potential for the equations (4.14)-(4.16). From (4.14)-(4.16) it follows that for a constant electric field $\vec{E} = -\nabla \phi$ and, consequently, ϕ obeys the equation

$$\Delta \phi = -4\pi a^2 \rho.$$

The invariant of the group of rotations $O(4)$ on a three-dimensional sphere of radius a is either the arc length or the angle between radius-vectors x and Y (here considered as corresponding to two S^3 tops),

$$X = (x^1, x^2, x^3, x^4), \ Y = (y^1, y^2, y^3, y^4), \ cos\theta = \frac{1}{a^2}(x^1y^1 + x^2y^2 + x^3y^3 + x^4y^4).$$

Since

$$M_{ij}\, cos\theta = (x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}) cos\theta = \frac{1}{a^2}(x^i y^j - x^j y^i),$$

setting in (4.18) $\rho = 0$, $\phi = \phi(z)$, where $z = cos\theta$, we obtain the equation for $\phi(z)$ [21]

$$(1 - z^2)\frac{d^2\phi}{dz^2} - 3z\frac{d\phi}{dz} = 0.$$

The general solution to this equation is of the form

$$\phi(z) = c_1 \frac{z}{\sqrt{1 - z^2}} + c_2 = c_1 \cot\theta + c_2,$$

where c_l and c_2 are arbitrary constants.
Introduce the frame of reference with respect to which one of the charged tops is at rest and has the coordinates $(0, 0, 0, -R)$. In this system consider a stereographic projection of the three-dimensional sphere (4.10) from point $(0, 0, 0, R)$ onto the hyperplane $x^4 = 0$ with Cartesian coordinates x, y, z. We have

$$x^1 = x\tfrac{2R^2}{r^2+R^2}, \quad x^2 = y\tfrac{2R^2}{r^2+R^2} \quad x^3 = z\tfrac{2R^2}{r^2+R^2}, \quad x^4 = R\tfrac{r^2-R^2}{r^2+R^2},$$

where $r^2 = x^2 + y^2 + z^2$. It may be verified that in the coordinates x, y, z

$$\cot\theta = \frac{R}{2r} - \frac{r}{2R} \tag{4.19}$$

and, consequently, the Coulomb potential on the S^3 can be written in the following form

$$\phi(r) = e(\frac{1}{r} - \frac{r}{R^2} - \frac{1}{R}). \tag{4.20}$$

We complete this section with an important remark+, that the potential (4.20) coincides with the known Cornell (potential, which has been defined empirically and successfully applied to the description of the mass spectrum of the charmonium [25].

5. Equations in Spherical Coordinates

Let us rewrite equation (4.11) in spherical coordinates. We are looking for spinorial wave functions of the following form

$$\Phi_{jm}(r,\theta,\psi) = \begin{pmatrix} \phi^+(r)Y^+_{jm}(\theta,\psi) + i\phi^-(r)Y^-_{jm}(\theta,\psi) \\ \Psi^+(r)Y^+_{jm}(\theta,\psi) + i\Psi^-(r)Y^-_{jm}(\theta,\psi) \end{pmatrix} \tag{5.1}$$

Inserting (5.1) into (4.11) and using

$$\frac{(\vec{\sigma}\vec{N})}{R} = \frac{\hbar}{i}[(1+\tfrac{r^2}{R^2})\tfrac{d}{dr} + \tfrac{(1+k)}{r}]\frac{(\vec{\sigma}\vec{r})}{r},$$

$$(\vec{\sigma}\vec{M})Y^{\pm}_{jm} = \hbar(J^2 - L^2 - \tfrac{3}{4})Y^{\pm}_{jm} = -\hbar(1+k)Y^{\pm}_{jm},$$

$$Y^{\pm}_{jm} = \frac{(\vec{\sigma}\vec{r})}{r}Y^{\pm}_{jm},$$

$$k = \begin{cases} -(\ell+1), \ j = \ell + 1/2 \\ +\ell, \ j = \ell - 1/2, \end{cases}$$

we get

$$(\mathcal{E} - mc)\phi^+ = \tfrac{(\ell+1)}{R}\Psi^+ + [(1+\tfrac{r^2}{R^2})\tfrac{d}{dr} - \tfrac{\ell-1}{r}]\Psi^-,$$

$$(\mathcal{E} + mc)\Psi^- = -\tfrac{(\ell-1)}{R}\phi^- - [(1+\tfrac{r^2}{R^2})\tfrac{d}{dr} + \tfrac{\ell+1}{r}]\phi^+,$$

$$(\mathcal{E} - mc)\phi^- = -\tfrac{(\ell-1)}{R}\Psi^- - [(1+\tfrac{r^2}{R^2})\tfrac{d}{dr} + \tfrac{\ell+1}{r}]\Psi^+,$$

$$(\mathcal{E} + mc)\Psi^+ = \frac{(\ell+1)}{R}\phi^+ + [(1+\frac{r^2}{R^2})\frac{d}{dr} - \frac{\ell-1}{r}]\phi^-, \tag{5.2}$$

It is more convenient to use the trigonometrical coordinates defined by

$$\tan x = \tfrac{r}{R}, \quad \sin x = (r/R)\sqrt{1 + r^2/R^2}.$$

In this notations eq. (5.2) becomes

$$\mathcal{E}^- \sin x\phi^+ = (\ell+1)\sin x\,\Psi^+ + (\sin x\tfrac{d}{dx} - (\ell-1)\cos x)\Psi^-,$$

$$\mathcal{E}^+ \sin x\Psi^- = -(\ell-1)\sin x\,\phi^- - (\sin x\frac{d}{dx} + (\ell+1)\cos x)\phi^+, \tag{5.3}$$

$$\mathcal{E}^- \sin x\phi^- = -(\ell-1)\sin x\,\Psi^- - (\sin x\tfrac{d}{dx} + (\ell+1)\cos x)\Psi^+,$$

$$\mathcal{E}^+ \sin x\Psi^+ = (\ell+1)\sin x\,\phi^+ + (\sin x\tfrac{d}{dx} - (\ell-1)\cos x)\phi^-,$$

$$\mathcal{E}^{\pm} = \mathcal{E} \pm mc.$$

The solutions we are looking for have the form

$$\phi^+ = \sum_k a_k \sin^k x, \quad \Psi^+ = \sum_k b_k \sin^k x,$$

$$\phi^- = \sum_k g_k \cos x \sin^{k-1} x, \quad \Psi^- = \sum_k c_k \cos x \sin^{k-1} x.$$

Substituting these series into eq.(5.3) and equating the expressions with sin^k and cos^k we obtain a system of algebraic equations for the coefficients a, b, g, c:

$$\mathcal{E}^- a_{k-1} = (\ell+1)b_{k-1} + (k-\ell+1)c_{k+1} - (k-\ell)c_{k-1},$$

$$\mathcal{E}^+ b_{k-1} = (\ell+1)a_{k-1} + (k-\ell+1)g_{k+1} - (k-\ell)g_{k-1},$$

$$\mathcal{E}^- g_k = -(\ell-1)c_k - (k+\ell+1)b_k,$$

$$\mathcal{E}^+ c_k = -(\ell-1)g_k - (k+\ell+1)a_k. \tag{5.4}$$

Presenting this system in matrix form

$$(k+\ell)(k-\ell+1)\begin{pmatrix} g_{k+1} \\ c_{k+1} \end{pmatrix} = (\hat{A})\begin{pmatrix} c_{k-1} \\ g_{k-1} \end{pmatrix},$$

where the matrix $(\hat{A})$

$$(\hat{A}) = \begin{pmatrix} \mathcal{E}^-\mathcal{E}^+ + 1 - k^2 & -2\mathcal{E}^- \\ -2\mathcal{E}^+ & \mathcal{E}^-\mathcal{E}^+ + 1 - k^2 \end{pmatrix}.$$

To obtain the regular solutions we equal the determinant of this matrix to zero:

$$Det(\hat{A}) = Det\begin{pmatrix} \mathcal{E}^-\mathcal{E}^+ + 1 - k^2 & -2\mathcal{E}^- \\ -2\mathcal{E}^+ & \mathcal{E}^-\mathcal{E}^+ + 1 - k^2 \end{pmatrix} = 0.$$

The energy spectrum coincides with (4.14)

$$\frac{\mathcal{E}^2}{c^2} = m^2c^2 + (n+1)^2/R^2.$$

Introduce now the Coulomb potential . For that purpose it is enough to substitute $\mathcal{E}^\pm \sin x + \frac{\alpha}{R}\cos x$ for $\mathcal{E}^\pm \sin x$, where $\alpha = e^2/(\hbar \cdot c)$. In that case we look for solutions of the form

$$\phi^+ = exp(-Dx)\sum_k [a_k sin^{k+s}x + A_k \cos x sin^{k-1+s}x],$$

$$\Psi^+ = exp(-Dx)\sum_k [b_k sin^{k+s}x + B_k \cos x sin^{k-1+s}x],$$

$$\phi^- = exp(-Dx)\sum_k [g_k \cos x sin^{k-1+s}x + G_k sin^{k+s}x],$$

$$\Psi^- = exp(-Dx)\sum_k [c_k \cos x sin^{k-1+s}x + E_k sin^{k+s}x]$$

Inserting these functions and equate the expressions in sin^k and cos^k to get for the coefficients the system of equations

$$\mathcal{E}^- a_k - \frac{\alpha}{R^2} A_k = \frac{\ell+1}{R} b_k - \frac{k+s-\ell}{R} c_k - \frac{D}{R} E_k -$$

$$-\frac{\alpha}{R^2} A_{k+2} + \frac{k+s+\ell+1}{R} c_{k+2},$$

$$\mathcal{E}^+ b_k - \frac{\alpha}{R^2} B_k = \frac{\ell+1}{R} a_k - \frac{k+s-\ell}{R} g_k - \frac{D}{R} G_k -$$

$$-\frac{\alpha}{R^2} B_{k+2} + \frac{k+s+\ell+1}{R} g_{k+2},$$

$$\frac{\mathcal{E}^-}{R} G_k - \frac{\alpha}{R} g_k = -\frac{\ell-1}{R^2} E_k + \frac{k+s+\ell}{R^2} B_k + D b_k -$$

$$-\frac{a}{R} g_{k+2} - \frac{k+s-\ell-1}{R^2} B_{k+2},$$

$$\frac{\mathcal{E}^+}{R} E_k - \frac{\alpha}{R} c_k = -\frac{\ell-1}{R^2} G_k + \frac{k+s+\ell}{R^2} A_k + D a_k -$$

$$-\frac{\alpha}{R} c_{k+2} - \frac{k+s-\ell-1}{R^2} A_{k+2},$$

$$\mathcal{E}^- g_k + \frac{\alpha}{R^2} G_k = -\frac{\ell-1}{R} c_k - \frac{k+s+\ell+1}{R} b_k + \frac{D}{R} B_k,$$

$$\mathcal{E}^+ c_k + \frac{\alpha}{R^2} E_k = -\frac{\ell-1}{R} g_k - \frac{k+s+\ell+1}{R} a_k + \frac{D}{R} A_k,$$

$$\frac{\mathcal{E}^-}{R} A_k + \frac{\alpha}{R} a_k = \frac{\ell+1}{R^2} B_k + \frac{k+s-\ell+1}{R^2} E_k - D\, c_k,$$

$$\frac{\mathcal{E}^+}{R} B_k + \frac{\alpha}{R} b_k = \frac{\ell+1}{R^2} A_k + \frac{k+s-\ell+1}{R^2} G_k - D\, g_k.$$

For the regular solutions the determinant of the system above must be zero

$$\mathcal{E}^- a_k - \frac{\alpha}{R^2} A_k - \frac{\ell+1}{R} b_k + \frac{k+s-\ell}{R} c_k + \frac{D}{R} E_k = 0,$$

$$\mathcal{E}^{+}b_k - \frac{\alpha}{R^2}B_k - \frac{\ell+1}{R}a_k + \frac{k+s-\ell}{R}g_k + \frac{D}{R}G_k = 0,$$

$$\frac{\mathcal{E}^{-}}{R}G_k - \frac{\alpha}{R}g_k + \frac{\ell-1}{R^2}E_k - \frac{k+s+\ell}{R^2}B_k - Db_k = 0,$$

$$\frac{\mathcal{E}^{+}}{R}E_k - \frac{\alpha}{R}c_k + \frac{\ell-1}{R^2}G_k - \frac{k+s+\ell}{R^2}A_k - Da_k = 0,$$

$$\mathcal{E}^{-}g_k + \frac{\alpha}{R^2}G_k + \frac{\ell-1}{R}c_k + \frac{k+s+\ell+1}{R}b_k - \frac{D}{R}B_k = 0,$$

$$\mathcal{E}^{+}c_k + \frac{\alpha}{R^2}E_k + \frac{\ell-1}{R}g_k + \frac{k+s+\ell+1}{R}a_k - \frac{D}{R}A_k = 0,$$

$$\frac{\mathcal{E}^{-}}{R}A_k + \frac{\alpha}{R}a_k - \frac{\ell+1}{R^2}B_k - \frac{k+s-\ell+1}{R^2}E_k + D\,c_k = 0,$$

$$\frac{\mathcal{E}^{+}}{R}B_k + \frac{\alpha}{R}b_k - \frac{\ell+1}{R^2}A_k - \frac{k+s-\ell+1}{R^2}G_k + D\,g_k = 0.$$

Acknowledgments

We thank the support of the National Research System of Mexico and of the Catedras of FES-C, U.N.A.M. system.

References

[1] Weyl H., *Zeitschrift Phys.*, **56** 330 (1929)
[2] Kustaanheimo P., E. Stiefel, *Journ. f. reine u. angew. Math.*, Berlin, **218** 204 (1965)
[3] Tait W., J. F. Cornwell, *Lett. Nuovo Cim.*, **3** 511 (1972)
[4] Yamaleev R. M. Comm. of JINR, P2-88-10, Dubna (1988)
[5] Penrose R., *J. of Math. Phys.* **8** 345 (1967)
[6] Penrose R., in: "Quantum Theory and Beyond" Ted Bastin, ed., Cambridge University Press, 1971
[7] Penrose R. and W. Reindler, Spinors and Space-Time, Cambridge Monographs on Mathematical Physics, Cambridge,
Vol. 1: Two spinor calculus and relativistic fields (1984);
Vol. 2: Spinor and twistor methods in space-time geometry (1986).
[8] Penrose R. and M. A. H. MacCallum, Twistor theory: An approach to the quantization of fields and space-time. *Phys. Rep.*, **6** 241 (1972)
[9] Ward R. S. and R. O. Wells, Twistor Geometry and Field Theory, Cambridge University Press, Cambridge (1990) and references therein.
[10] Hestenes D., Spacetime Algebra, Gordon and Breach, (1966)
[11] J. Keller, A. Rodriguez, R. Yamaleev *Advances in Applied Clifford Algebras* **6** (2), (1996) 275-300; Keller J. and F. Megy, La Teoría del Spin en Mecánica Cuántica no Relativista, *Contactos*, **1** (1) 51-54 (1984).

[12] Liu Yu-Fen and J. Keller , A Symmetry of Massless Fields, *J. Math. Phys.* **37** (9), 4320-4332 (1996)

[13] Keller J., Twistors as Geometric Objets in Spacetime, in: “Clifford Algebras and Spinor Structures”, P. Lounesto y R. Abłamowicz eds., Kluwer Academic Publishers, 133-136 (1995); Twistors and Clifford Algebras, in: “Clifford Algebras and their Applications in Mathematical Physics”, V. Dietrich, K. Habetha and G. Jank, eds., Kluwer Academic Publishers, 161-173 (1998).

[14] Keller J., Spinors, twistors, screws, mexors and the massive spinning electron in: “The Theory of the Electron”, J. Keller, Z. Oziewicz eds., *Advances in Applied Clifford Algebras* **7** (S), 439-455 (1997); Keller J., Spinors and Multivectors as a Unified Tool for Spacetime Geometry and for Elementary Particle Physics, *Int. Journal of Theoretical Physics* **30** (2), 137-184 (1991).

[15] Yamaleev R. M. Comm. of JINR, E2-84-197, Dubna (1984)

[16] Schrödinger E., *Proc. Roy. Irish. Acad.* **A 46**, 9 (1940)

[17] Yamaleev R. M. Comm. of JINR, P2-84-727, Dubna (1984)

[18] Proca A. Compt. Rend. 202, 1420 (1936);
Young J. A., S. A. Bludman, *Phys. Rev.* **131** (5), 2326 (1963)

[19] Kähler E., Die Dirac gleichung, *Abh. Dt. Akad. Wiss. Berlin Kl. für Math., Phys., und Techn.* N1 (1962)

[20] Pestov A. B. Comm. of JINR, P2-12886, Dubna (1979)

[21] Pestov A. B. *Hadronic J.* **17** (6), 603-614 (1994)

[22] Yamaleev R. M. Comm. of JINR, P2-85-722, Dubna (1985)

[23] Dirac P. A. M., Wave equations in de Sitter space, *Ann. of Math.* **36** (3), 657 (1935)

[24] Dirac P. A. M., Wave equations in Conformal space, *Ann. of Math.* **37** (2), 429 (1936); Gursey F., in: “Relativity, Groups and Topology” C. De Witt, B. De Witt, eds., New York-London, 1964; see also [12].

[25] Godfrey S., N. Isgur, *Phys. Rev.* **D 32**, 189 (1985)
Rabin J. M., *Phys. Rev.* **D 24**, 3218 (1981);
Susskind L., *Phys. Rev.* **D 16**, 3031 (1977)
Becher P., *Phys. Lett.* **D 104**, 221 (1988)