THE ELECTROMAGNETIC POTENTIAL AMONG NONRELATIVISTIC ELECTRONS

Gu Ying-Qiu
Institute of Mathematics,
Fudan University Shanghai,
200433, China

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Abstract. In this paper, we derive the second order nonrelativistic approximation of the spinor equations with electromagnetic interaction. This approximation gives the total Breit and Coulomb potential among the electrons. The derivation also shows that the nonlinear coupled spinor equation can give a more natural description for fundamental particles, and quantum theories seem to be computing methods for solving the dynamical equations of Hamiltonian system.

1. Introduction

According to quantum electrodynamics (QED), G. Breit and S. N. Gupta drived the $O(\frac{1}{m^2})$ order nonrelativistic approximation of the electromagnetic potential between two interacting electrons [1]. In what follows, we will show how the mutually coupled spinor equations also implies the Breit potential. This derivation and the analysis in [2,3] reveal that QED is a pretending form of the superposition principle and integral transformations for numerical computation, so it might be more appropriate to regard QED as a computing method for solving a special kind of dynamical equations for many-body problem, rather than a physical theory with new content.

This study is a development of some preceding papers [4-10]. By a general framework [2-4], we find that it is much convenient to describe the Fermions such as electrons by nonlinear coupled spinor equations, rather than by one. Apart from concise and concrete, this description can really lead to some unusual results, such as the natural combination of particle and wave [4,5], the

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complete Newtonian mechanics for many-body [2,6], the perfect fluid model in general relativity [7,8], the singularity-free Universe [9,10], etc.

2. The Nonrelativstic Electromagnetic Potential

Denote $\alpha^\mu=(I,\alpha^1,\alpha^2,\alpha^3)$, where

$$\alpha_k=\begin{pmatrix}0 & \sigma_k\\ \sigma_k & 0\end{pmatrix},\quad (k=1,2,3),\quad \gamma_0=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix},$$

σ_k is the Pauli's matrix. The space-time metric is defined as

$$(\eta^{\mu\nu})=\mathrm{diag}(-1,1,1,1).$$

In [2-8], we find that the following dynamical equation system can give a better description for N electrons moving in the external potential Φ_μ,

$$\begin{cases}(i\partial_0-e\sum_{l\neq k}A_{0,l}-e\Phi_0)\phi_k=\vec{\alpha}\cdot(\vec{p}_k+e\sum_{l\neq k}\vec{A}_l+e\vec{\Phi})\phi_k+m\gamma_0\phi_k,\\ \Box A_l^\mu=eq_l^\mu\equiv-e\phi_l^+\alpha^\mu\phi_l,\quad \partial_\mu A_l^\mu=0,\end{cases}\tag{2.1}$$

where we have chosen the natural unit $\hbar=c=1$, $\phi_k(k=1,2,\cdots N)$ is a bispinor to describe the electron k, $\vec{p}_k\equiv-i\nabla-e\vec{A}_E$, the subscript E stands for external fields. In (2.1) we have merged the self-interaction into the physical mass m. For each ϕ_k, making Foldy-Wouthuysen transformation [11,12] and keeping $\mathrm{O}(\frac{1}{m^2})$ terms, we have

$$\begin{aligned}i\partial_t\psi_k=\ &\hat{H}_k\psi_k+\frac{4\pi}{e}\alpha\sum_{l\neq k}\left(V_{0,l}-\frac{1}{m}(\vec{V}_l\cdot\vec{p}_k+\vec{s}_k\cdot\vec{B}_l)\right)\psi_k\\ &-\frac{\alpha\pi}{em^2}\left(\vec{\sigma}\cdot(\vec{E}_l\times\vec{p}_k-\frac{1}{2}i\nabla\times\vec{E}_l)+\frac{1}{2}\nabla\cdot\vec{E}_l\right)\psi_k,\end{aligned}\tag{2.2}$$

where we have omitted the retarded potential of each spinor, $\alpha=\dfrac{e^2}{4\pi}$, $\vec{s}_k\equiv\dfrac{1}{2}\vec{\sigma}$ stands for the spin of the electron k,

$$\vec{V}_l=\int\frac{e\phi_l^+\vec{\alpha}\phi_l}{4\pi|\vec{x}-\vec{X}_l|}d^3X_l\doteq\frac{e}{8\pi m}\int\frac{(\vec{\sigma}\cdot\vec{p}_l\psi_l)^+\vec{\sigma}\psi_l+\psi_l^+\vec{\sigma}(\vec{\sigma}\cdot\vec{p}_l\psi_l)}{|\vec{x}-\vec{X}_l|}d^3X_l,$$

$$V_{0,l}=\int\frac{e|\phi_l|^2}{4\pi|\vec{x}-\vec{X}_l|}d^3X_l\doteq\int\frac{e|\psi_l|^2}{4\pi|\vec{x}-\vec{X}_l|}d^3X_l,\quad \vec{E}_l=-\nabla V_{0,l},\quad \vec{B}_l=\nabla\times\vec{V}_l,$$

$$\begin{aligned}\hat{H}_k=\frac{1}{2m}\vec{p}_k^2+\ e\Phi_0-\frac{e}{2m}\vec{\sigma}\cdot\vec{B}_E-\frac{e}{4m^2}\Big(\vec{\sigma}\cdot(\vec{E}_E\times\vec{p}_k\\ -\frac{1}{2}i\nabla\times\vec{E}_E)+\frac{1}{2}\nabla\cdot\vec{E}_E\Big).\end{aligned}\tag{2.3}$$

The total energy of the system can be expressed by

$$H=\sum_{k=1}^N H_k+\frac{1}{2}\alpha\left(P_0-\frac{1}{m^2}(P_1+P_2+P_3+P_4)\right),\tag{2.4}$$

where each term is defined and calculated as follows. Denote $r_{kl}=|\vec{X}_k-\vec{X}_l|$, $\vec{L}_{kl}\equiv(\vec{X}_k-\vec{X}_l)\times\vec{p}_k$. By using the following formula

$$\begin{cases}(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B})=\vec{A}\cdot\vec{B}+i\vec{\sigma}\cdot(\vec{A}\times\vec{B}),\quad \nabla\frac{\vec{r}}{r^3}=-\nabla^2\frac{1}{r}=4\pi\delta^3(r),\\ (\vec{A}\times\vec{\sigma})(\vec{B}\cdot\vec{\sigma})=\vec{A}\times\vec{B}-i(\vec{A}\cdot\vec{B})\vec{\sigma}+i(\vec{A}\cdot\vec{\sigma})\vec{B},\\ (\vec{B}\cdot\vec{\sigma})(\vec{A}\times\vec{\sigma})=\vec{A}\times\vec{B}+i(\vec{A}\cdot\vec{B})\vec{\sigma}-i(\vec{A}\cdot\vec{\sigma})\vec{B},\end{cases}$$

$$\vec{s}\cdot\nabla_l\left(\vec{\sigma}\cdot\nabla_l\frac{1}{r_{kl}}\right)=-\frac{1}{r_{kl}}\vec{s}\cdot\vec{\sigma}+\frac{3}{r_{kl}^5}(\vec{s}\cdot\vec{r}_{kl})(\vec{\sigma}\cdot\vec{r}_{kl})-\frac{4\pi}{3}\vec{s}\cdot\vec{\sigma}\delta^3(r_{kl}),$$

we have

$$H_k\equiv\int_{R^3}\psi_k^+H_k\psi_k d^3X_k=\int_{R^{3N}}\Psi^+H_k\Psi d^{3N}X=\langle|H_k|\rangle,\tag{2.5}$$

where $\Psi\equiv\psi_1(t,\vec{X}_1)\psi_2(t,\vec{X}_2)\cdots\psi_N(t,\vec{X}_N)$ is just an algebraic notation similar to $|1,2,\cdots,N\rangle$ in the occupation number representation, but if we represent all spins by $\vec{s}_k=\lambda_k\vec{n}_k$, $(\forall k,\lambda_k=\pm\frac{1}{2},\ |\vec{n}_k|=1)$, then one of the components of $\psi_k(\forall k)$ vanishes and Ψ becomes the common products of ψ_k. In (2.5) and the following equations, we have used the normalizing conditions

$\int_{R^3} |\psi_k(t, X_k)|^2 d^3X_k = 1, (\forall k)$, and assigned the dummy coordinate $\vec{X}_k$ to the k-th electron.

$$P_0 \equiv \frac{4\pi}{e}\sum_{k=1}^{N}\sum_{l\neq k}\int \psi_k^+ V_{0,l}\psi_k d^3X_k = \sum_{k,l\neq k}\int \Psi^+\frac{1}{r_{kl}}\Psi d^{3N}X = \sum_{l\neq k}\langle|\frac{1}{r_{kl}}|\rangle, \tag{2.6}$$

$$\begin{aligned} P_1 &\equiv \frac{4\pi m}{e}\sum_{k,l\neq k}\int \psi^+\vec{V}_l(X_k)\cdot\vec{p}_k\psi_k d^3X_k \\ &= \frac{1}{2}\sum_{k,l\neq k}\int \psi_k^+\left(\int\frac{1}{r_{kl}}[(\vec{\sigma}\cdot\vec{p}_l\psi_l)^+\vec{\sigma}\psi_l + \psi_l^+\vec{\sigma}(\vec{\sigma}\cdot\vec{p}_l\psi_l)]d^3X_l\right)\cdot\vec{p}_k\psi_k d^3X_k \\ &= \frac{1}{2}\sum\int\psi_k^+\psi_l^+\left(\frac{2}{r_{kl}}\vec{p}_k\cdot\vec{p}_l - \frac{i}{r_{kl}^3}\vec{\sigma}\cdot(\vec{X}_k-\vec{X}_l)\vec{\sigma}\cdot\vec{p}_k\right)\ \psi_l\psi_k d^3X_l d^3X_k \\ &= \sum_{l\neq k}\langle|\frac{1}{r_{kl}}\vec{p}_k\cdot\vec{p}_l - \frac{i}{2r_{kl}^3}(\vec{X}_k-\vec{X}_l)\cdot\vec{p}_k + \frac{1}{r_{kl}^3}\vec{s}_l\cdot\vec{L}_{kl}|\rangle. \end{aligned}$$

Since P_1 is a real, we have

$$P_1 = \frac{1}{2}(P_1^+ + P_1) = \sum_{l\neq k}\langle|\frac{1}{r_{kl}}\vec{p}_k\cdot\vec{p}_l + \pi\delta^3(r_{kl}) + \frac{1}{r_{kl}^3}\vec{s}_l\cdot\vec{L}_{kl}|\rangle. \tag{2.7}$$

Similarly, we have

$$\begin{aligned} P_2 &\equiv \frac{4\pi m}{e}\sum_{l\neq k}\int\psi_k^+\vec{s}_k\cdot(\nabla_k\times\vec{V}_l)\psi_k d^3X_k \\ &= \langle|\frac{1}{r_{kl}^3}\vec{s}_k\cdot(\vec{L}_{lk}-\vec{s}_l) + \frac{8\pi}{3}\vec{s}_k\cdot\vec{s}_l\delta^3(r_{kl}) \\ &\quad + \frac{3}{r_{kl}^5}(\vec{X}_k-\vec{X}_l)\cdot\vec{s}_k(\vec{X}_k-\vec{X}_l)\cdot\vec{s}_l|\rangle. \end{aligned} \tag{2.8}$$

$$P_3 \equiv 2\frac{\pi}{e}\sum_{l\neq k}\int\psi_k^+\vec{\sigma}\cdot(\vec{E}_l\times\vec{p}_k + \frac{i}{2}\nabla_k\times\vec{E}_l)\psi_k d^3X_k = \sum_{l\neq k}\langle|\frac{1}{r_{kl}^3}\vec{s}_k\cdot\vec{L}_{kl}|\rangle. \tag{2.9}$$

$$P_4 \equiv \frac{\pi}{e} \sum_{l \neq k} \int \psi_k^+ \frac{1}{2} \nabla_k \cdot \vec{E}_l \psi_k d^3 X_k = \sum_{l \neq k} \langle | \frac{\pi}{2} \delta^3(r_{kl}) | \rangle. \tag{2.10}$$

Substituting (2.5-10) into (2.4), we get the total Hamiltonian and Schrödinger eqution of the system

$$\hat{H} = \sum_{k=1}^{N} H_k + \frac{1}{2} \sum_{k=1}^{N} \sum_{l \neq k} P_{kl}, \quad i\partial_t \Psi = \hat{H} \Psi, \tag{2.11}$$

where P_{kl} is the total electromagnetic potential between electron k and electron l, including the Breit potential and Coulomb potential

$$\begin{aligned} P_{kl} = \; & \frac{\alpha}{r_{kl}} - \frac{\alpha}{m^2} \left(\frac{1}{r_{kl}} \vec{p}_k \cdot \vec{p}_l + \frac{3\pi}{2} \delta^3(r_{kl}) + \frac{8\pi}{3} \delta^3(r_{kl}) \vec{s}_k \cdot \vec{s}_l \right. \\ & \left. + \frac{1}{r_{kl}^3} \vec{s}_k \cdot (2\vec{L}_{lk} - \vec{s}_l + \vec{L}_{kl}) + \frac{3}{r_{kl}^5} (\vec{X}_k - \vec{X}_l) \cdot \vec{s}_k (\vec{X}_k - \vec{X}_l) \cdot \vec{s}_l \right). \end{aligned} \tag{2.12}$$

There is a little difference between (2.11-12) and the result of QED. In QED, the operator

$$\vec{x} \cdot (\vec{x} \cdot \vec{p}_2) \vec{p}_1 = \vec{x}^2 \; \vec{p}_1 \cdot \vec{p}_2 + (\vec{x} \times (\vec{x} \times \vec{p}_2)) \cdot \vec{p}_1$$

was not partially integrated, where $\vec{x} = \vec{X}_1 - \vec{X}_2$.

The above derivations are all in classical level and well defined, but similarly to [2-8], we always get the right results of the quantum theory more naturally and completely. In fact, after some analysis [2], we find that the Heisenberg representation and q-number interpretation are just approximation computing method compatible with the dynamical equation. The only essentially new concept of the quantum theory imposed on the classical field theory is the hypothesis that an initial state $|A\rangle$ evolving into a final state $|B\rangle$ is undetermined in some cases and the probability is in proportion to $\langle B|A\rangle^2$. However $\langle B|A\rangle$ is the projection of $|B\rangle$ on $|A\rangle$, and $1 - \langle B|A\rangle^2$ is a measurement of distance between the two vectors, so this interpretation is really reasonable in logic. This treatment can bring convenience in solving some complicated processes such as scattering, but a complete description must be the solution of coupled field dynamical equations.

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